Design and Analysis of Algorithms

CSE 5311

Lecture 2 Algorithms and Growth Functions

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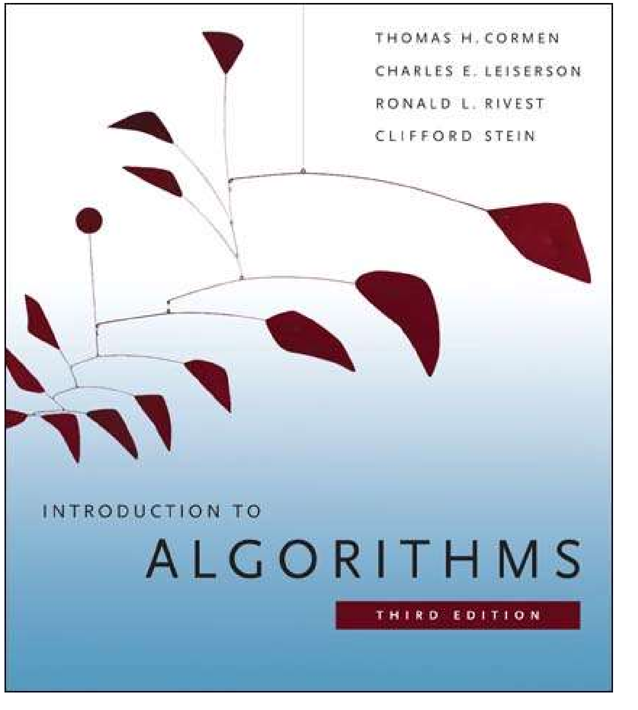
Department of Computer Science and Engineering

Administration

- **Course CSE5311**
  - What: Design and Analysis of Algorithms
  - When: Friday 1:00 ~ 3:50pm
  - Where: ERB 130
  - Who: Junzhou Huang (Office ERB 650) jzhuang@uta.edu
  - Office Hour: Friday 3:50 ~ 5:50pm and/or appointments
  - Homepage: http://ranger.uta.edu/~huang/teaching/CSE5311.htm
    (**You're required to check this page regularly**)
- **Lecturer**
  - PhD in CS from Rutgers, the State University of New Jersey
  - Research areas: machine learning, computer vision, medical image analysis and bioinformatics
- **GTA**
  - Saiyang Na (Office ERB 403), sxn3892@mavs.uta.edu
  - Office hours: Friday 10:00am ~ 12:00pm and/or appointments

Reviewing: Study Materials

- **Prerequisites**
  - Algorithms and Data Structure (CSE 2320)
  - Theoretical Computer Science (CSE 3315)
  - What this really means:
    - You have working experience s on software development.
    - You know compilation process and programming
    - Elementary knowledge of math and algorithms
- **Text book**
  - Thomas H. Cormen, Charles E. Leiserson, Ronald L. Rivest and Clifford Stein, **Introduction to Algorithms**, third edition
  - https://mitpress.mit.edu/books/introduction-algorithms



Reviewing: What?

- **The theoretical study of design and analysis of computer algorithms**
- **Basic goals for an algorithm**
  - Always correct
  - Always terminates
- **Our class: performance**
  - Performance often draws the line between what is possible and what is impossible.
- **Design and Analysis of Algorithms**
  - **Analysis:** predict the cost of an algorithm in terms of resources and performance
  - **Design:** design algorithms which minimize the cost

Reviewing: Insertion Sort

8 2 4 9 3 6

2 8 4 9 3 6

2 4 8 9 3 6

2 4 8 9 3 6

2 3 4 8 9 6

2 3 4 6 8 9 *done*

Reviewing: Running Time

- **Running Time**
  - Depends on the input
  - An already sorted sequence is easier to sort.
- **Major Simplifying Convention**
  - Parameterize the running time by the size of the input, since short sequences are easier to sort than long ones.
  - $T_A(n)$ = time of A on length n inputs. Generally, we seek upper bounds on the running time, to have a guarantee of performance.
- **Kinds of Analyses**
  - **Worst-case:** (usually) T(n) = maximum time of algorithm on any input of size n
  - **Average-case:** (sometimes) T(n) = expected time of algorithm over all inputs of size n. Need assumption of statistical distribution of inputs.
  - **Best-case:** (Never) Cheat with a slow algorithm that works fast on some input.

Machine-independent Time

- **Question**
  - Machine-independent Time
- **Idea**
  - Ignore machine dependent constants, otherwise impossible to verify and to compare algorithms
  - Look at growth of T(n) as $n \to \infty$.

"Asymptotic Analysis"

Recall: Θ-notation

Definition:

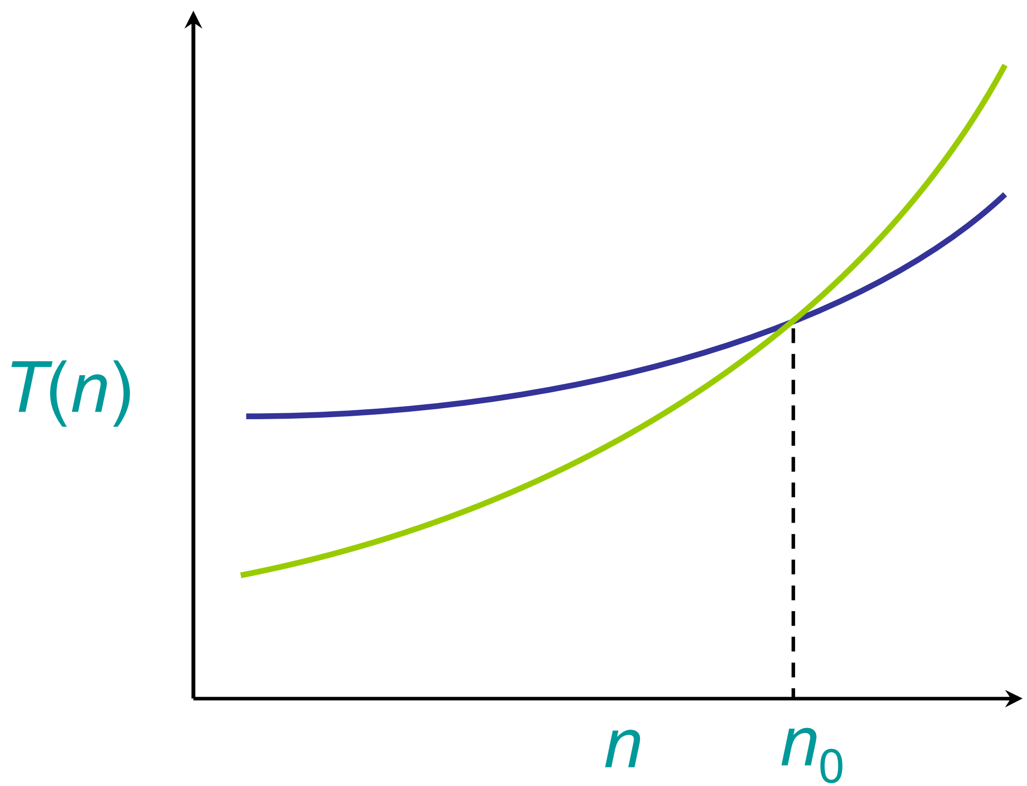
$\Theta(g(n)) = \{ f(n)$: there exist positive constants c_1, c_2, and n_0 such that $0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n)$ for all $n \geq n_0$ $\}$

Basic Manipulations:

- Drop low-order terms; ignore leading constants.
- Example: $3n^3 + 90n^2 - 5n + 6046 = \Theta(n^3)$

Asymptotic Performance

When n gets large enough, a $\Theta(n^2)$ algorithm *always* beats a $\Theta(n^3)$ algorithm.



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- Asymptotic analysis is a useful tool to help to structure our thinking toward better algorithm
- We shouldn't ignore asymptotically slower algorithms, however.
- Real-world design situations often call for a careful balancing

Insertion Sort Analysis

Worst case: Input reverse sorted.

$$T(n) = \sum_{j=2}^{n} \Theta(j) = \Theta(n^2)$$

[arithmetic series]

Average case: All permutations equally likely.

$$T(n) = \sum_{j=2}^{n} \Theta(j/2) = \Theta(n^2)$$

Is insertion sort a fast sorting algorithm?

- Moderately so, for small n.
- Not at all, for large n.

Integer Multiplication

- **Let X = $\boxed{A \mid B}$ and Y = $\boxed{C \mid D}$ where A,B,C and D are n/2 bit integers**
- **Simple Method:**

$$XY = (2^{n/2}A+B)(2^{n/2}C+D) = 2^n AC + 2^{n/2}(AD+BC) + BD$$

- **Running Time Recurrence**

$$T(n) < 4T(n/2) + \Theta(n)$$

Recursive Calls **Addition and Shift**

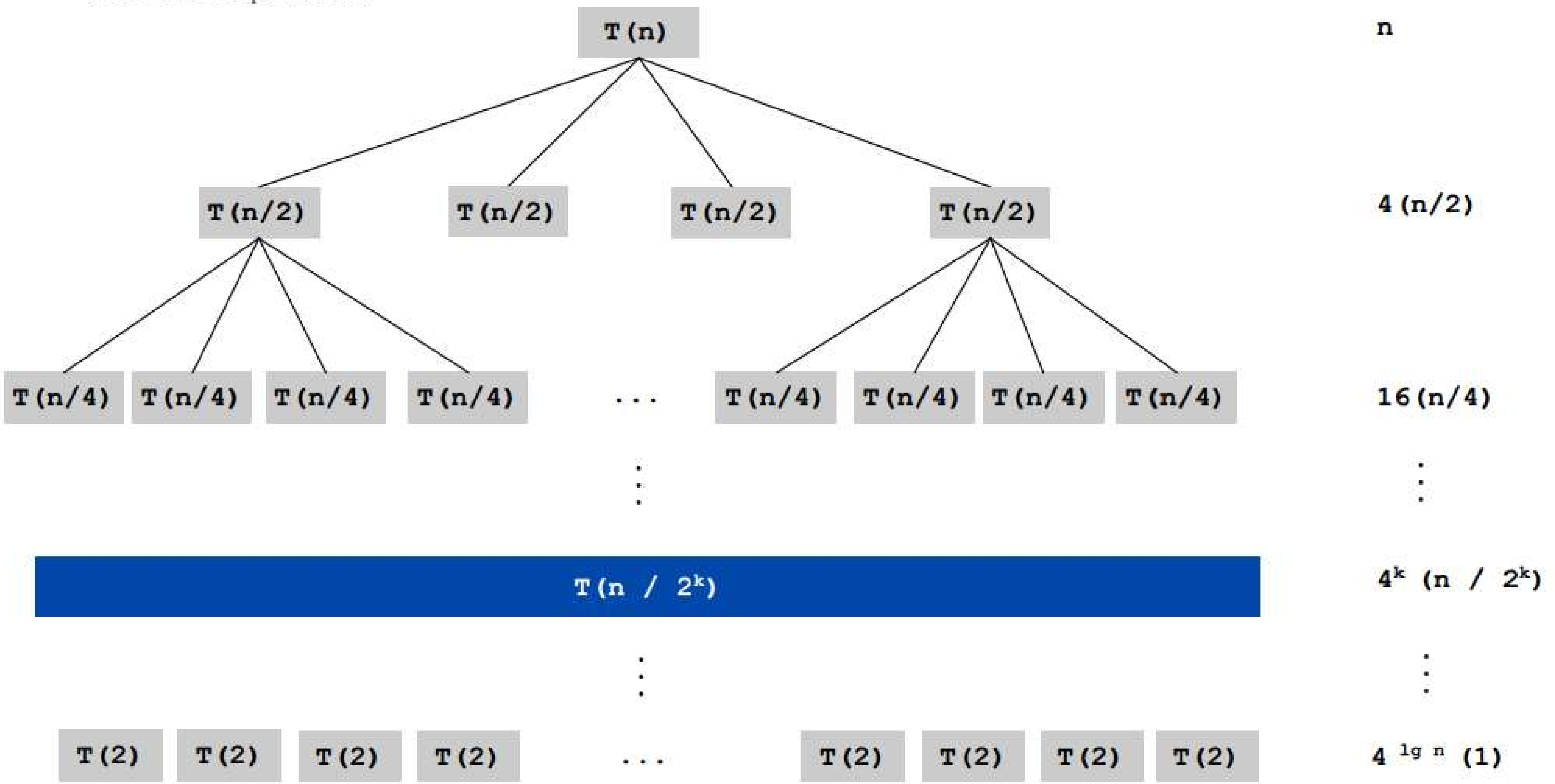
- **Solution $T(n) = \Theta(n^2)$**

Integer Multiplication

$$T(n) = \begin{cases} 0 & \text{if } n = 0 \\ 4T(n/2) + n & \text{otherwise} \end{cases}$$

assume n is a power of 2

$$T(n) = \sum_{k=0}^{\lg n} n\, 2^k = n\left(\frac{2^{1+\lg n} - 1}{2-1}\right) = 2n^2 - n$$



Better Integer Multiplication

- Let X = | A | B | and Y = | C | D | where A,B,C and D are n/2 bit integers
- [Karatsuba-Ofman 1962] : Can multiply two n-bit integers in $O(n^{\log 3})$ bit operations.

$$XY = (2^{n/2}A+B)(2^{n/2}C+D) = 2^{n}AC+2^{n/2}(AD+BC)+BD$$

$$= (2^{n} - 2^{n/2})AC+2^{n/2}(A+B)(C+D) + (1 - 2^{n/2})BD$$

- **Running Time Recurrence**

$$T(n) < 3T(n/2) + \Theta(n)$$

$$T(n) \le \underbrace{T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + T(1+\lceil n/2 \rceil)}_{\text{recursive calls}} + \underbrace{\Theta(n)}_{\text{add, subtract, shift}} \Rightarrow T(n) = O(n^{\lg 3}) = O(n^{1.585})$$

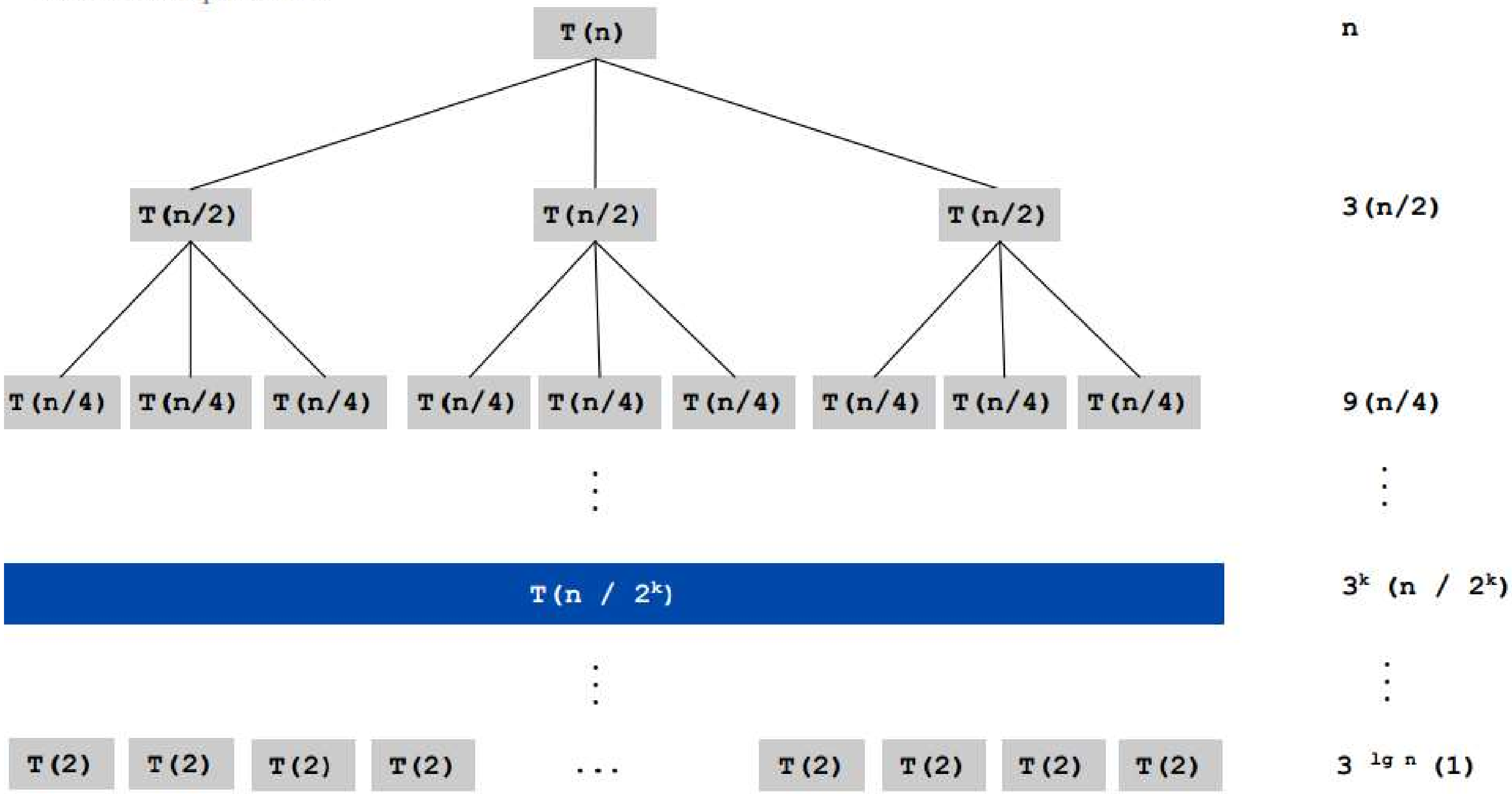
- **Solution: $T(n) = O(n^{\log 3})$**

Better Integer Multiplication

$$T(n) = \begin{cases} 0 & \text{if } n = 0 \\ 3T(n/2) + n & \text{otherwise} \end{cases}$$

assume n is a power of 2

$$T(n) = \sum_{k=0}^{\lg n} n \left(\tfrac{3}{2}\right)^k = n\left(\frac{\left(\frac{3}{2}\right)^{1+\lg n} - 1}{\frac{3}{2} - 1}\right) = 3n^{\lg 3} - 2n$$



Merge Sort

MERGE-SORT $A[1 \,..\, n]$

1. **If** $n = 1$**, done.**
2. **Recursively sort** $A[\,1 \,..\, \lceil n/2 \rceil\,]$ **and** $A[\,\lceil n/2 \rceil + 1 \,..\, n\,]$ **.**
3. **"*Merge*" the** 2 **sorted lists.**

Key subroutine: **MERGE**

Merging Two Sorted Arrays

20	12
13	11
7	9
2	1

Merging Two Sorted Arrays

20	12
13	11
7	9
2	1

1

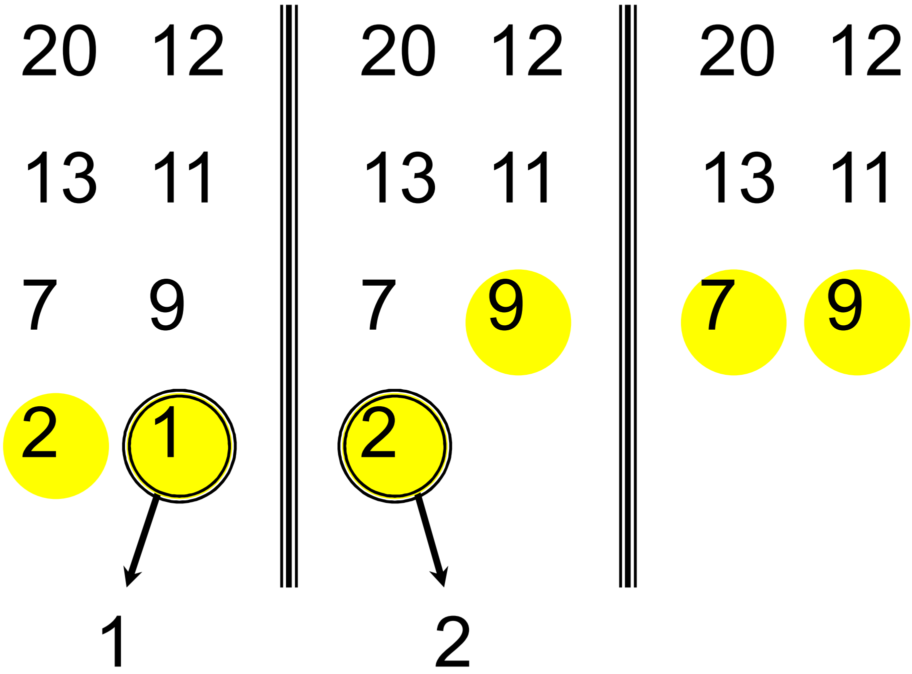
Merging Two Sorted Arrays

20	12	‖	20	12
13	11	‖	13	11
7	9	‖	7	9
2	1	‖	2	
	1			

Merging Two Sorted Arrays

20	12		20	12
13	11		13	11
7	9		7	9
2	1		2	
	1		2	

Merging Two Sorted Arrays



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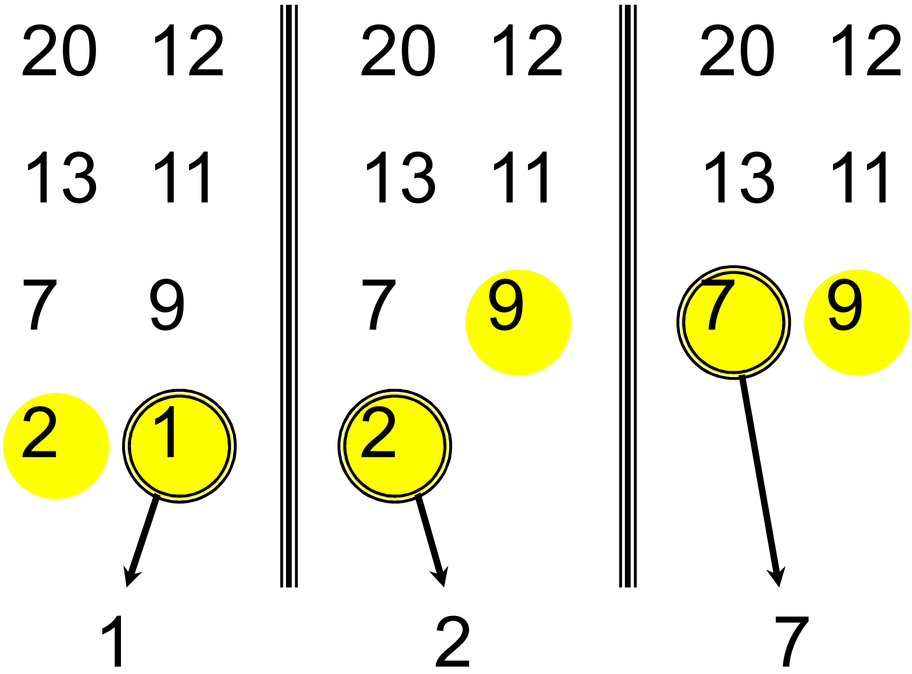

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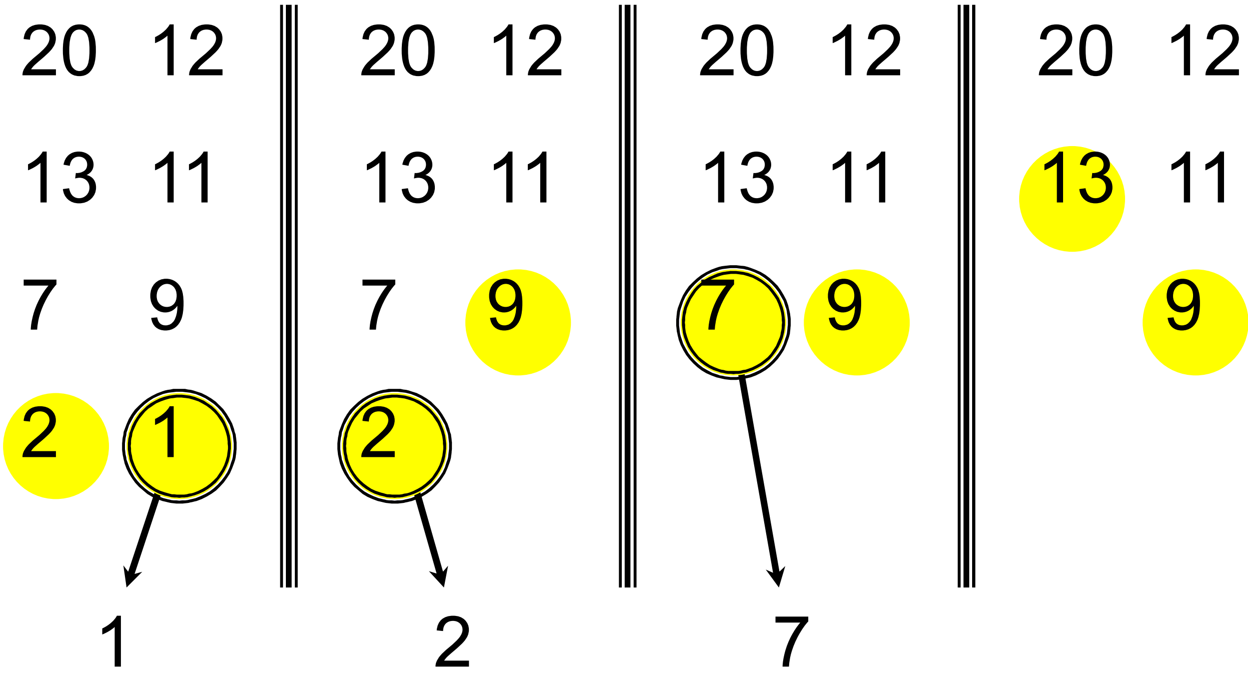

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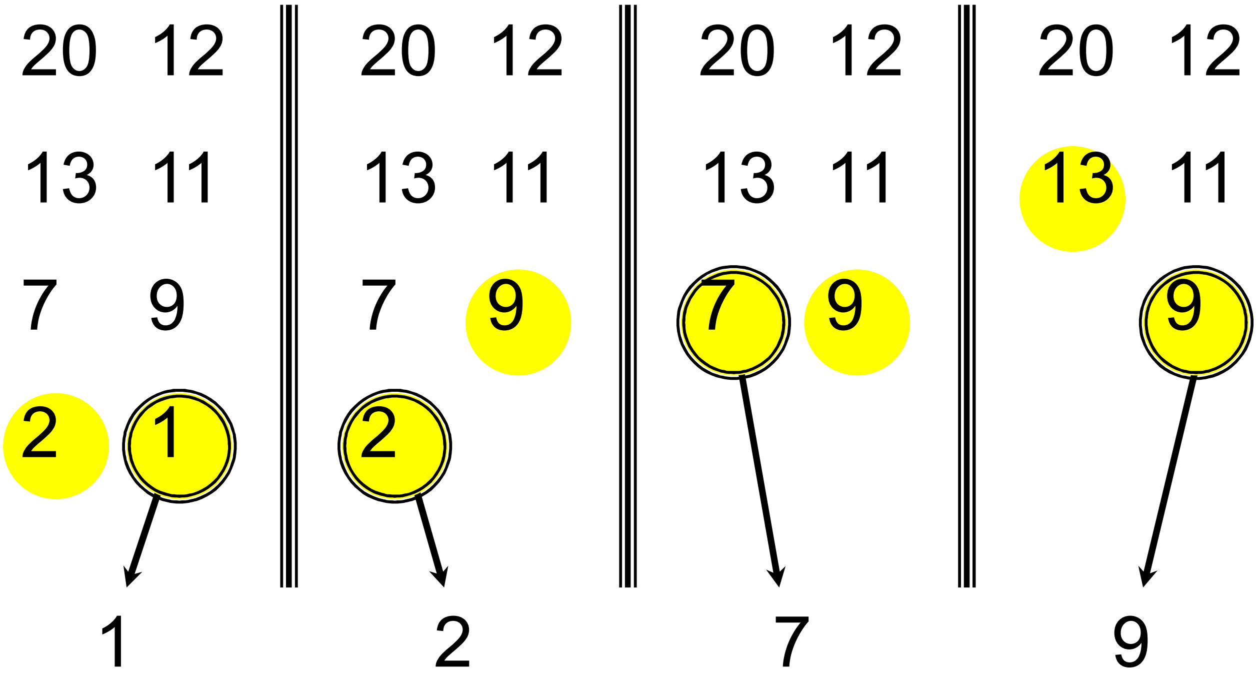

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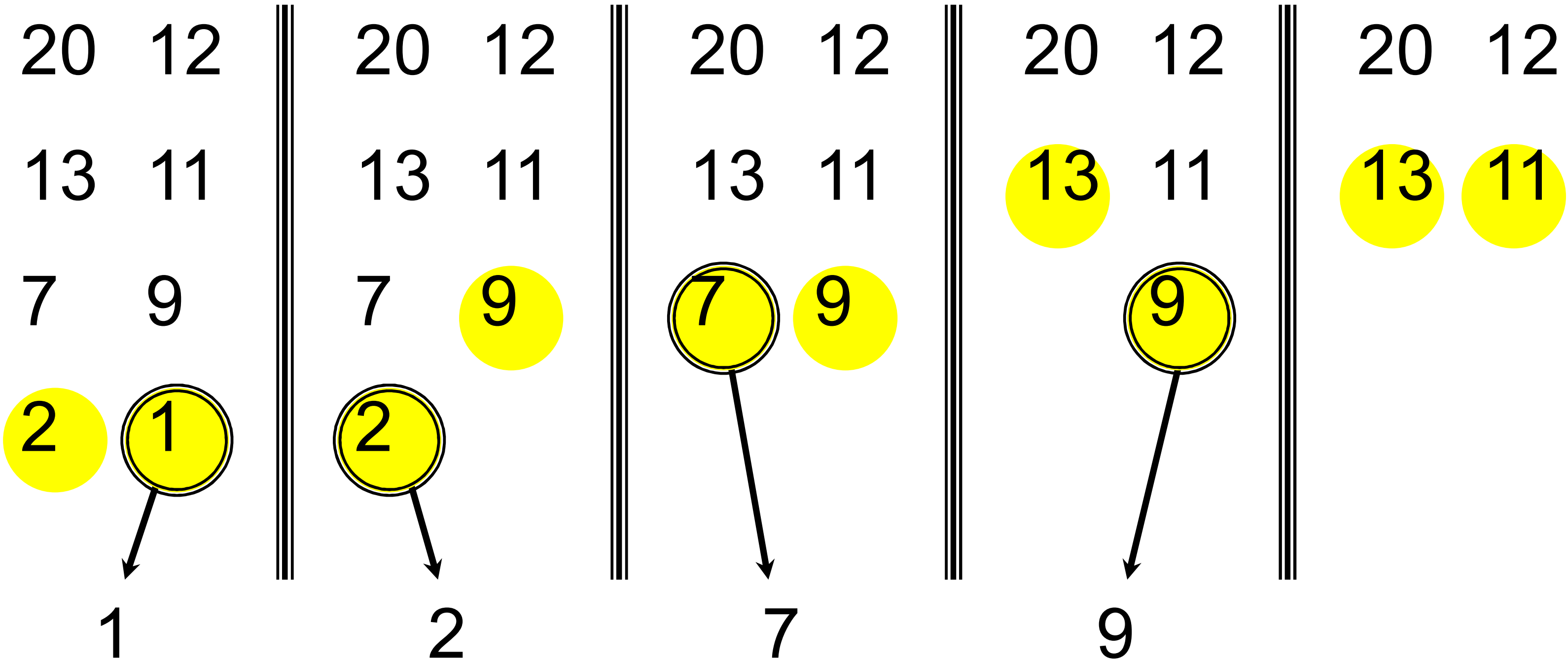

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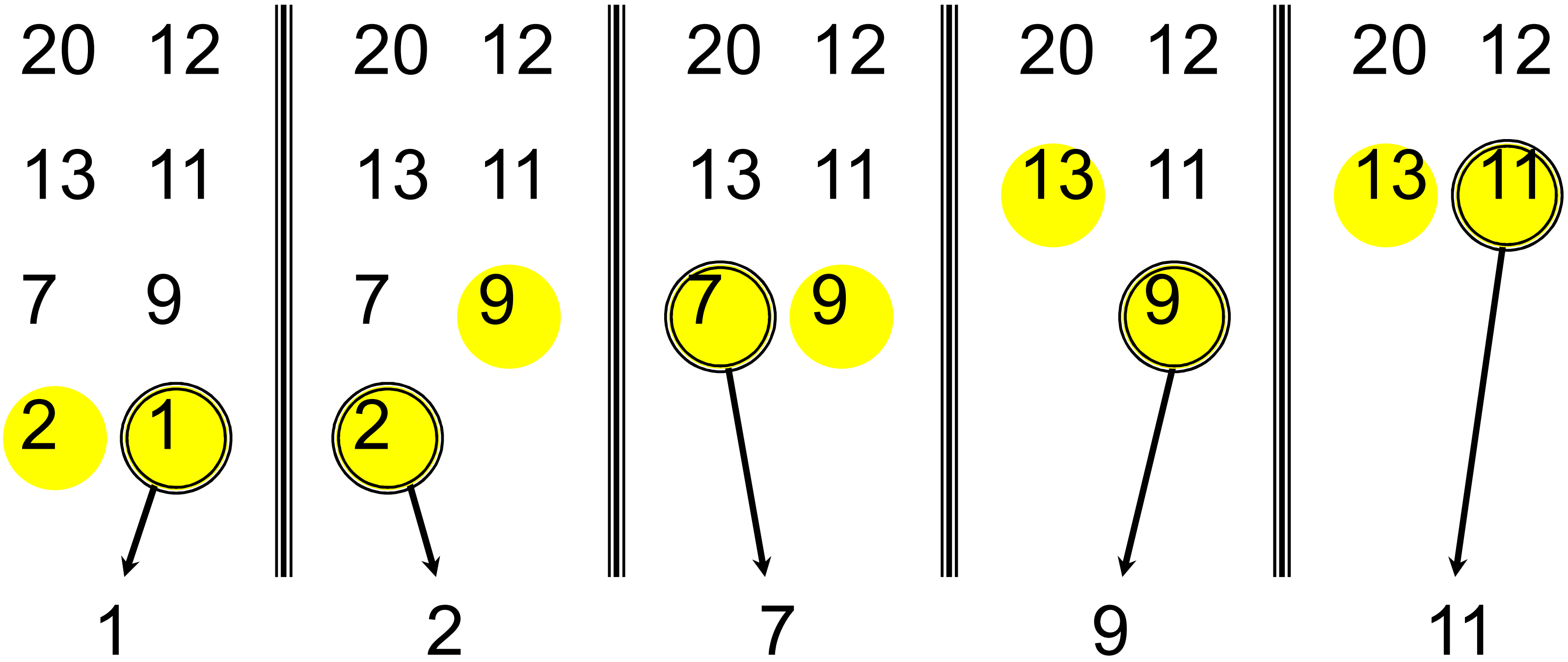

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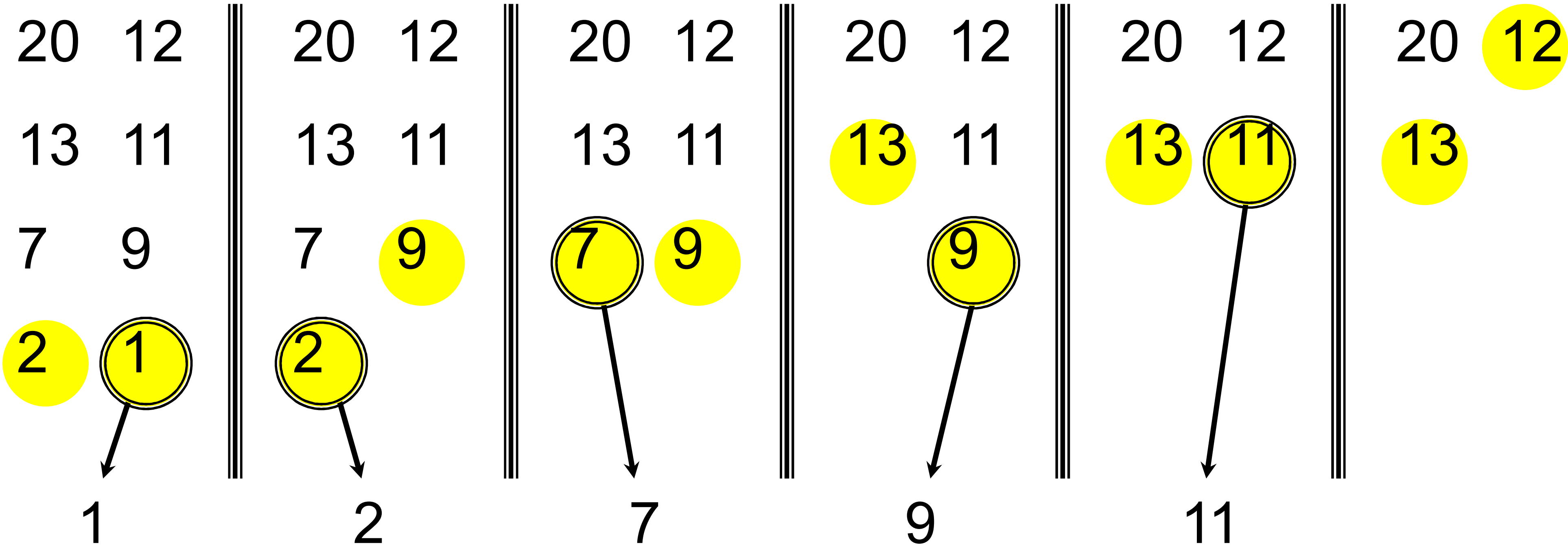

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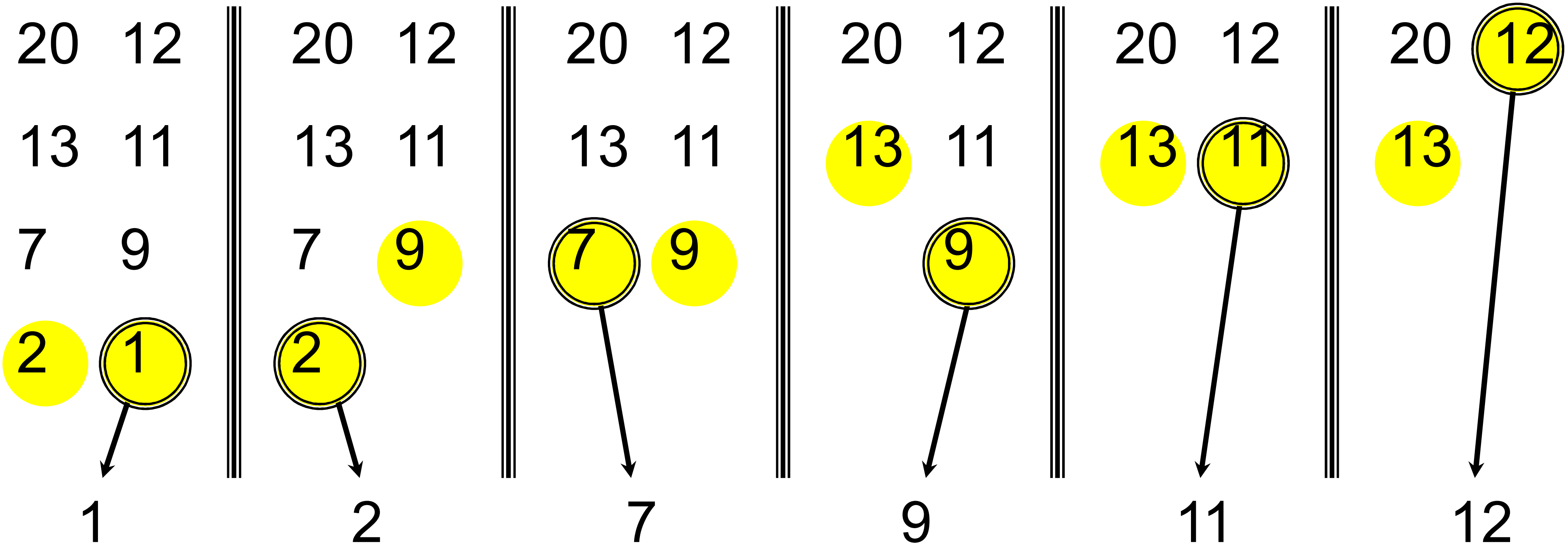

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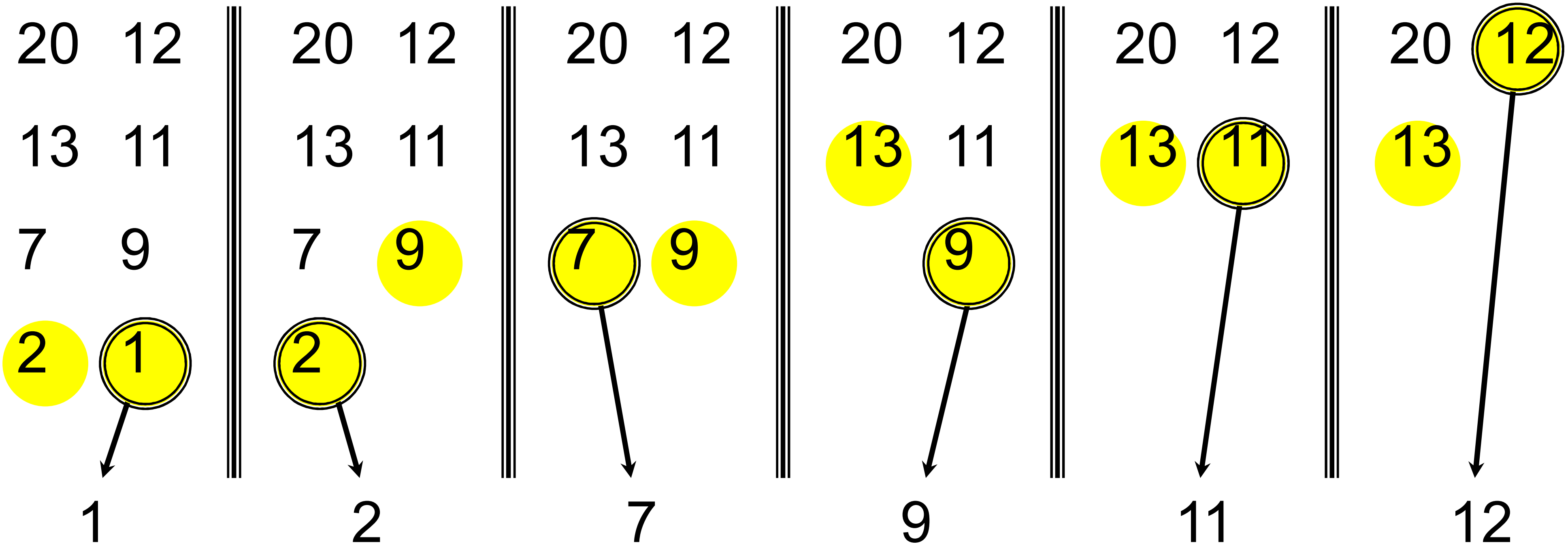

Time = $\Theta(n)$ to merge a total of n elements (linear time).

Analyzing Merge Sort

$T(n)$	**MERGE-SORT** $A[1 \mathinner{..} n]$
$\Theta(1)$	1. If $n = 1$, done.
$2T(n/2)$	2. Recursively sort $A[1 \mathinner{..} \lceil n/2 \rceil]$ and $A[\lceil n/2 \rceil + 1 \mathinner{..} n]$.
$\Theta(n)$	***3. "Merge"*** the 2 sorted lists

Sloppiness: **Should be** $T(\lceil n/2 \rceil) + T(\lfloor n/2 \rfloor)$ **, but it turns out not to matter asymptotically.**

Recurrence for Merge Sort

$$T(n) = \begin{cases} \Theta(1) \text{ if } n = 1; \\ 2T(n/2) + \Theta(n) \text{ if } n > 1. \end{cases}$$

- **We shall usually omit stating the base case when $T(n) = \Theta(1)$ for sufficiently small n, but only when it has no effect on the asymptotic solution to the recurrence.**
- **Next Lecture will provide several ways to find a good upper bound on $T(n)$.**

Recursion Tree

Solve $T(n) = 2T(n/2) + cn$, where $c > 0$ is constant.

$$cn$$

$$T(n/2) \qquad\qquad T(n/2)$$

Recursion Tree

Solve $T(n) = 2T(n/2) + cn$, where $c > 0$ is constant.

$$
\begin{array}{ccccccc}
 & & & cn & & & \\
 & cn/2 & & & & cn/2 & \\
T(n/4) & & T(n/4) & & T(n/4) & & T(n/4)
\end{array}
$$

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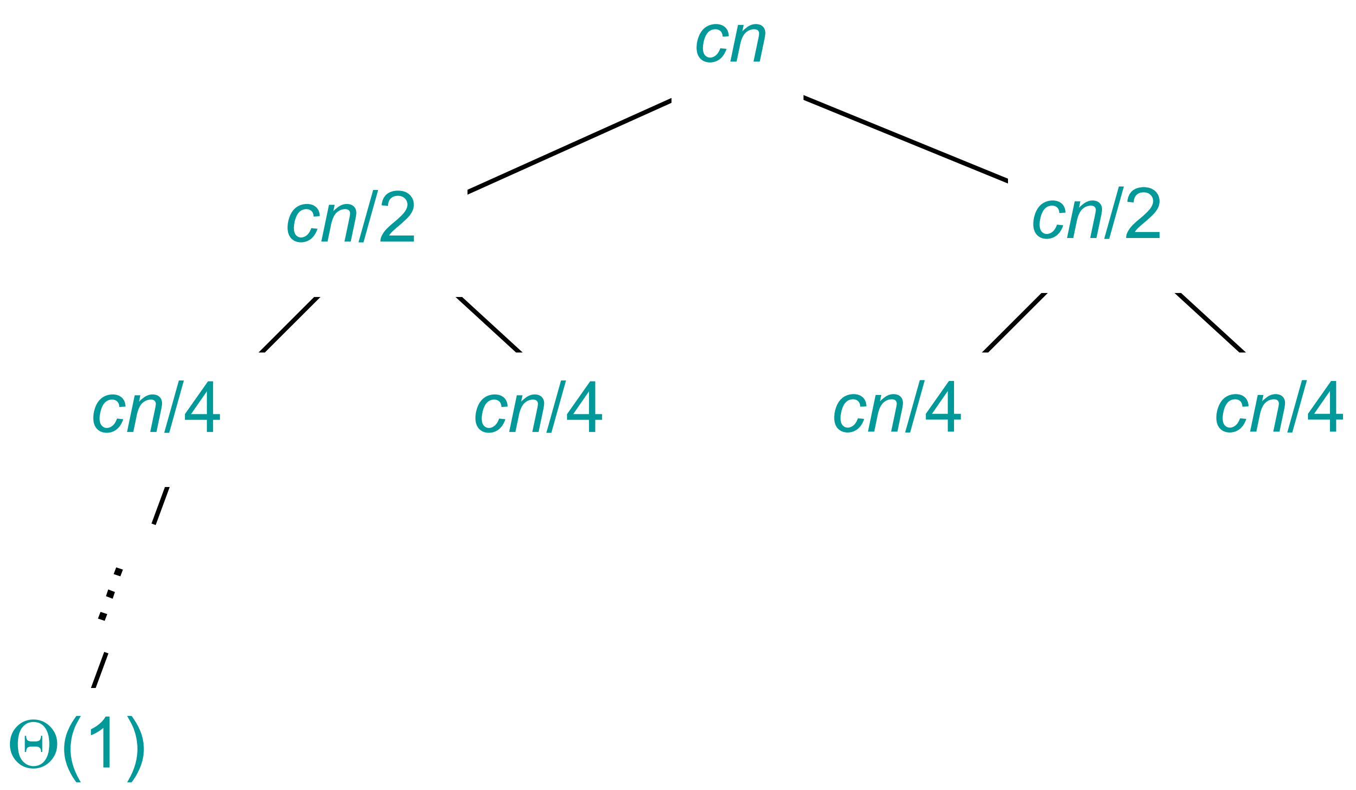

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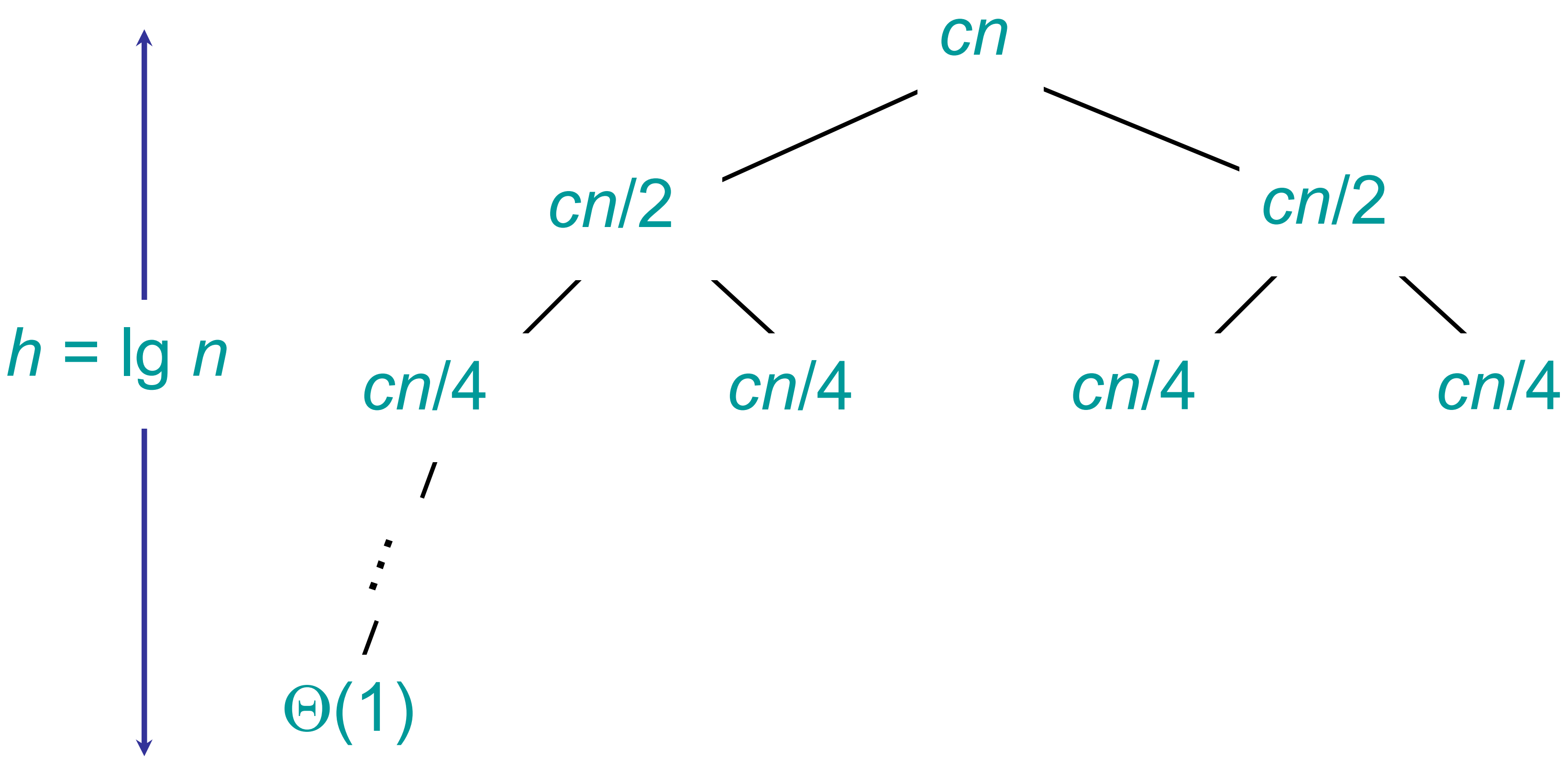

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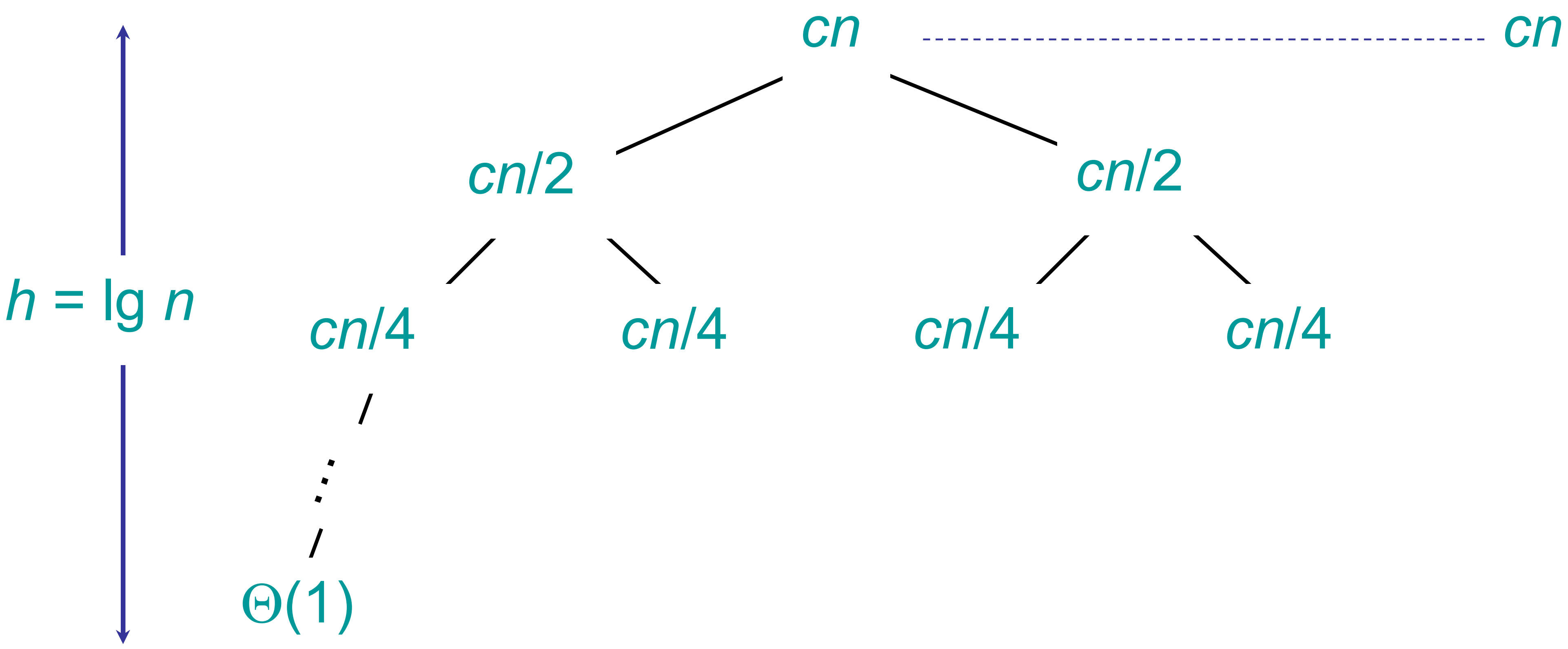

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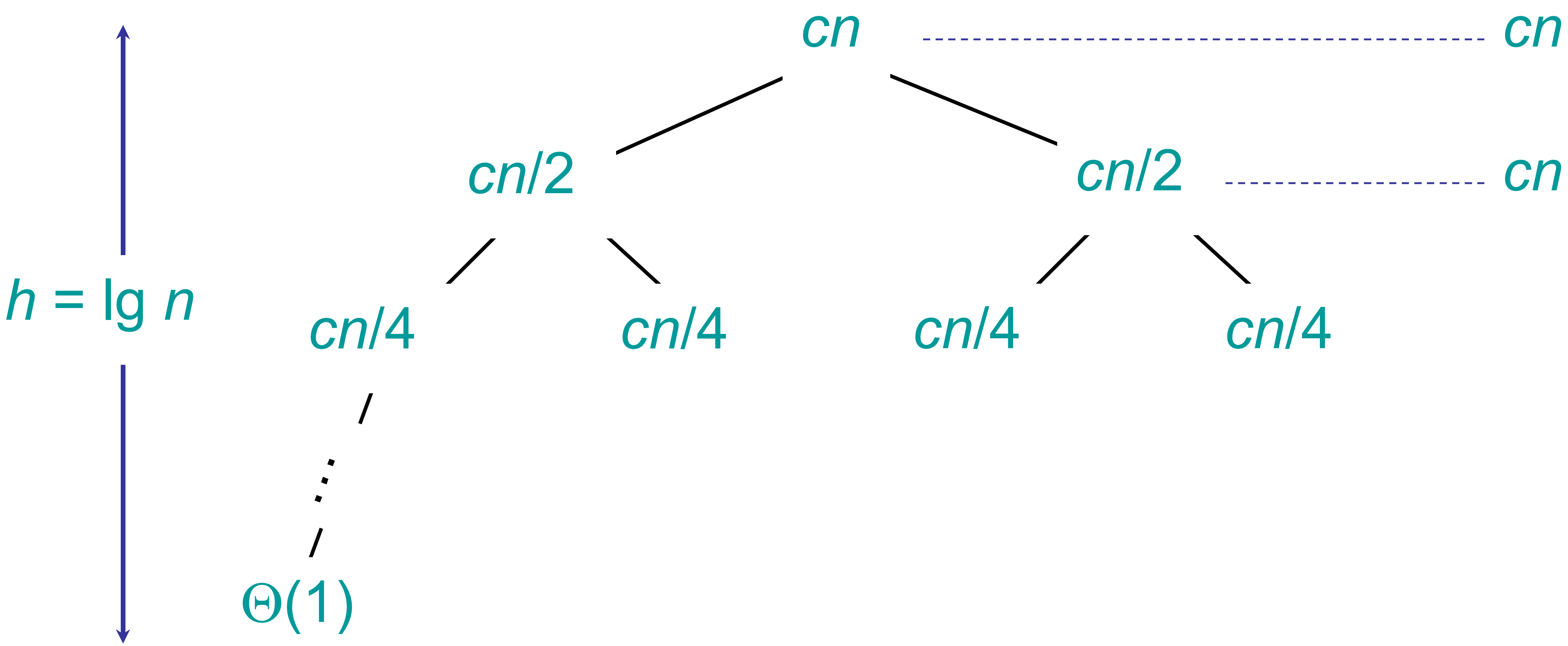

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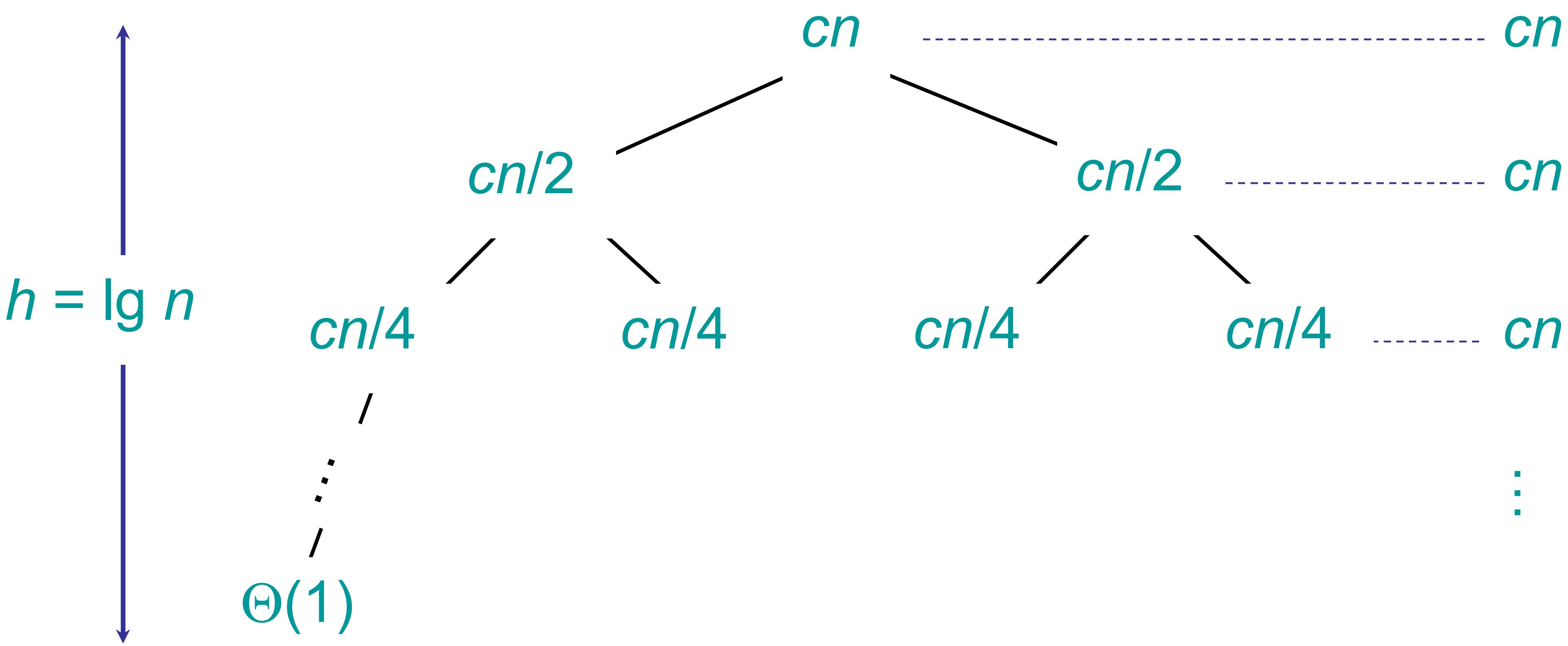

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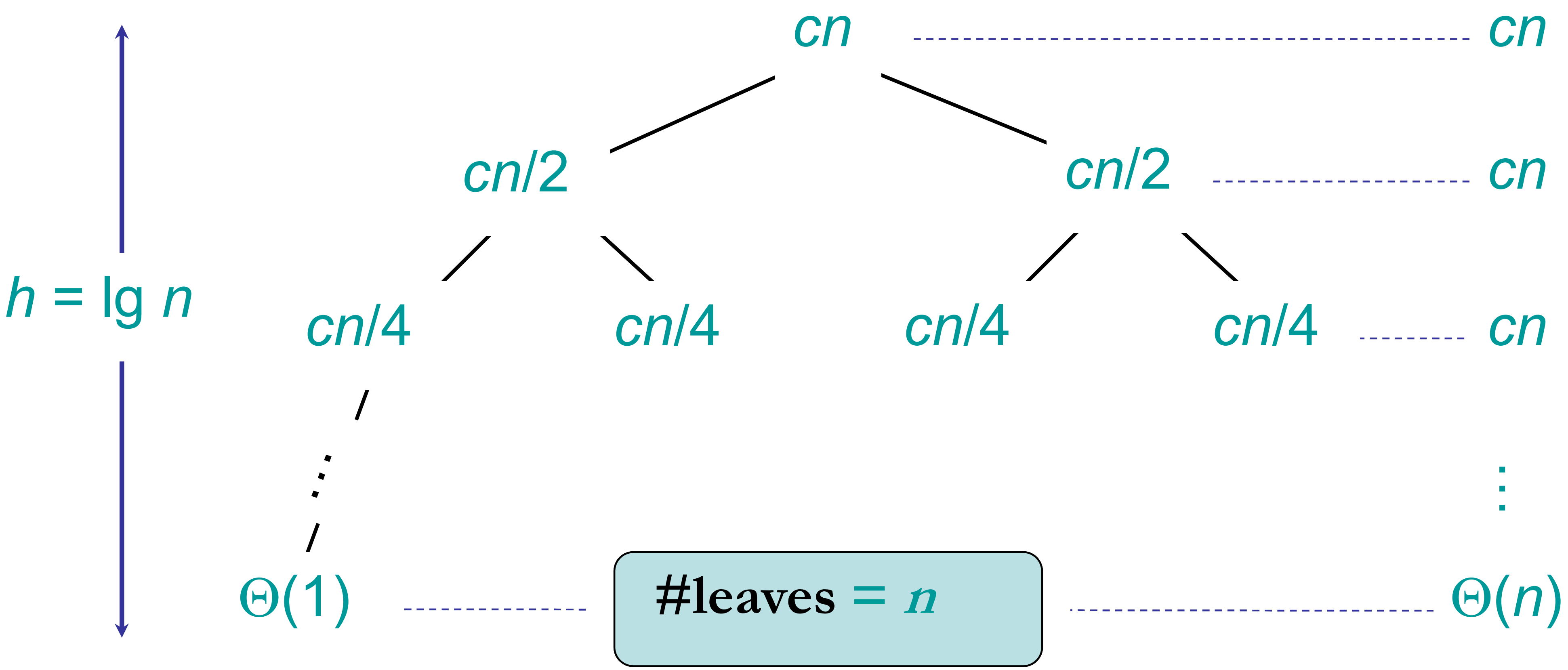

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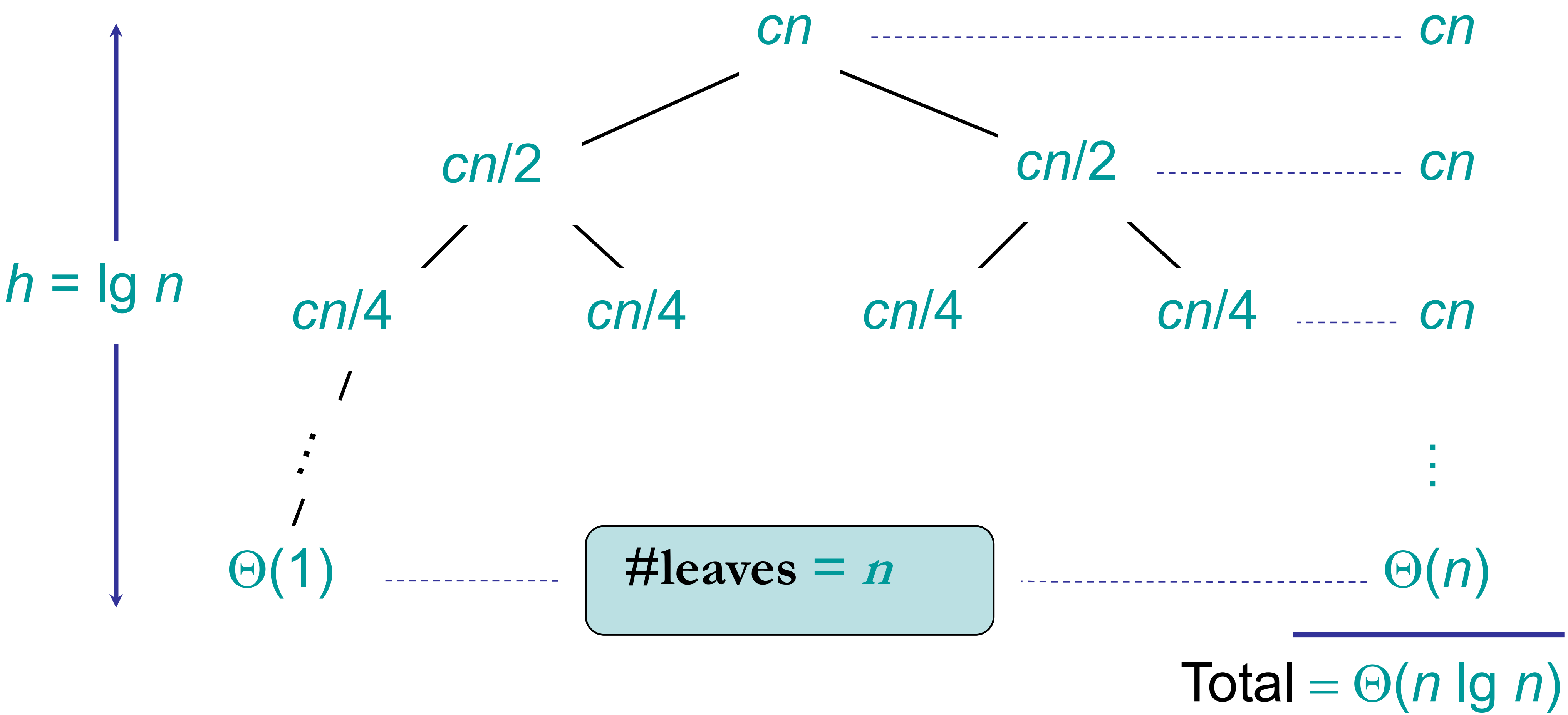

Summary

- $\Theta(n \lg n)$ grows more slowly than $\Theta(n^2)$.
- Therefore, merge sort asymptotically beats insertion sort in the worst case.
- In practice, merge sort beats insertion sort for $n > 30$ or so.