Design and Analysis of Algorithms

CSE 5311

Lecture 7 Quick Sort

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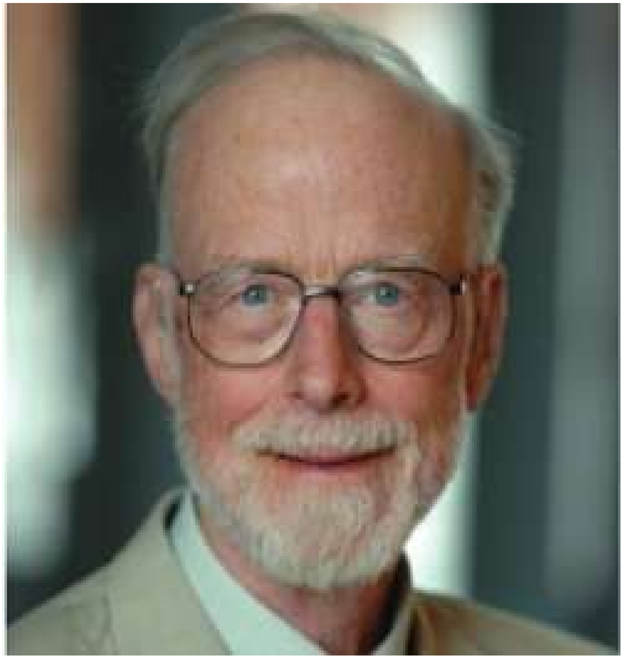
Quicksort

- Proposed by C.A.R. Hoare in 1962.
- Divide-and-conquer algorithm.
- Sorts "in place" (like insertion sort, but not like merge sort).
- Very practical (with tuning).



Divide and Conquer

Quicksort an n-element array:

1. ***Divide:*** Partition the array into two subarrays around a ***pivot*** x such that elements in lower subarray $\leq x \leq$ elements in upper subarray.

$\leq x$	x	$\geq x$

2. ***Conquer:*** Recursively sort the two subarrays.
3. ***Combine:*** Trivial.

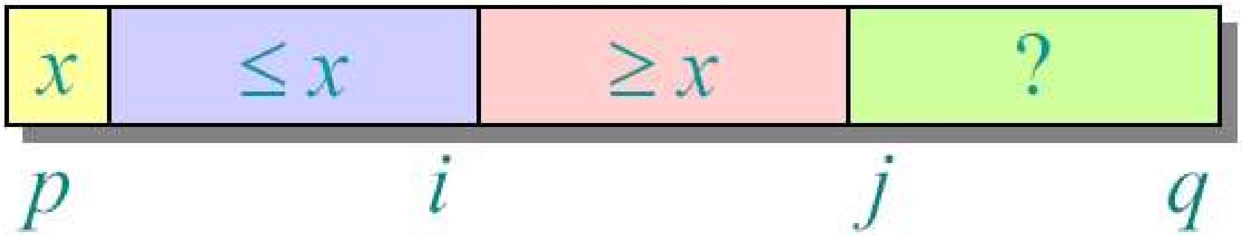
Key: *Linear-time partitioning subroutine.*

Partitioning Subroutine

PARTITION(A, p, q) ▷ $A[p \,.\,.\, q]$
    $x \leftarrow A[p]$ ▷ pivot $= A[p]$
    $i \leftarrow p$
    for $j \leftarrow p + 1$ **to** q
        do if $A[j] \leq x$
            then $i \leftarrow i + 1$
                exchange $A[i] \leftrightarrow A[j]$
    exchange $A[p] \leftrightarrow A[i]$
    return i

Running time $= O(n)$ for n elements.

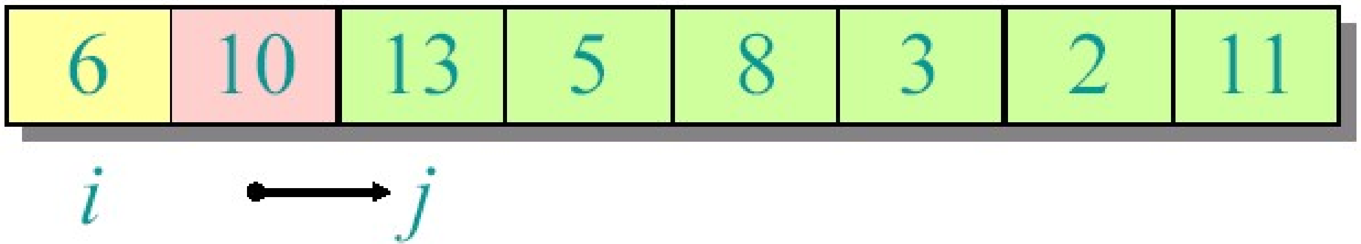
Invariant:



Example of Partitioning

6	10	13	5	8	3	2	11
i	j						

Example of Partitioning



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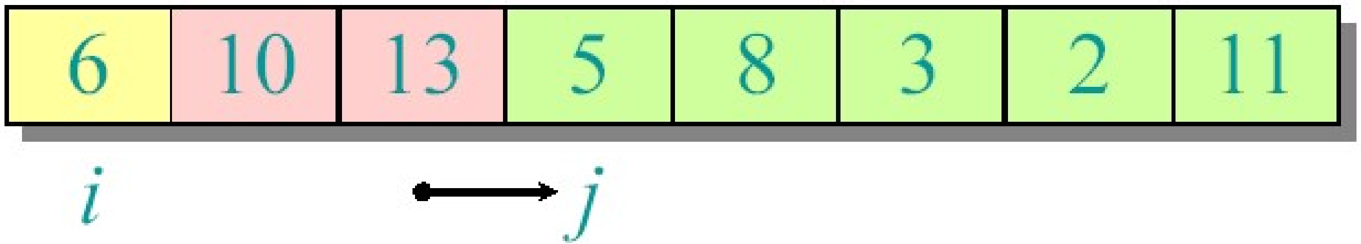

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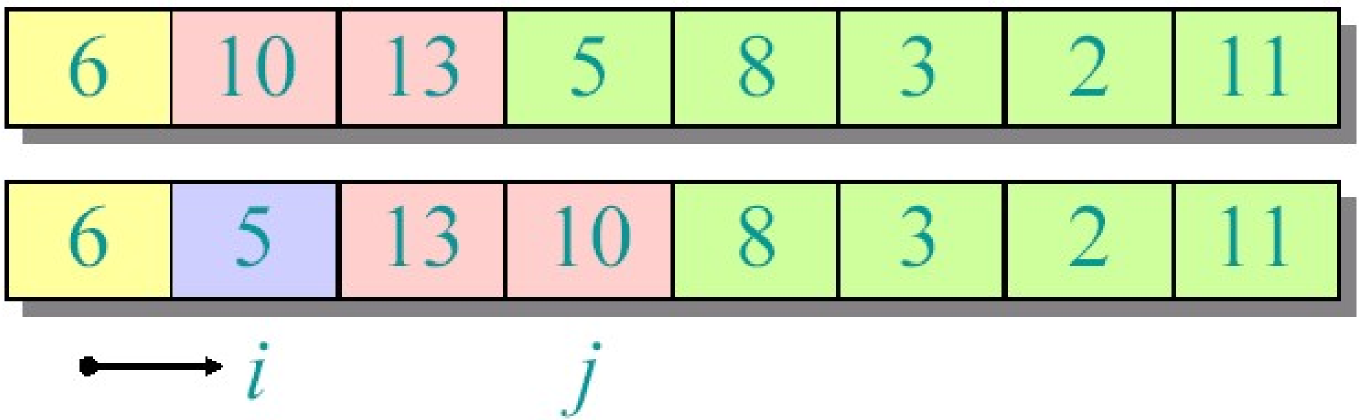

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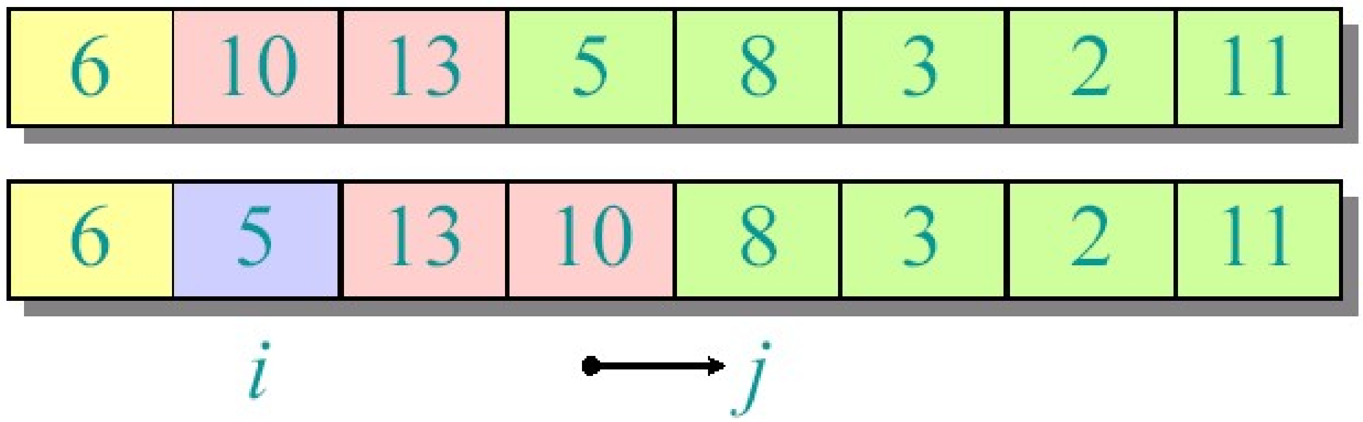

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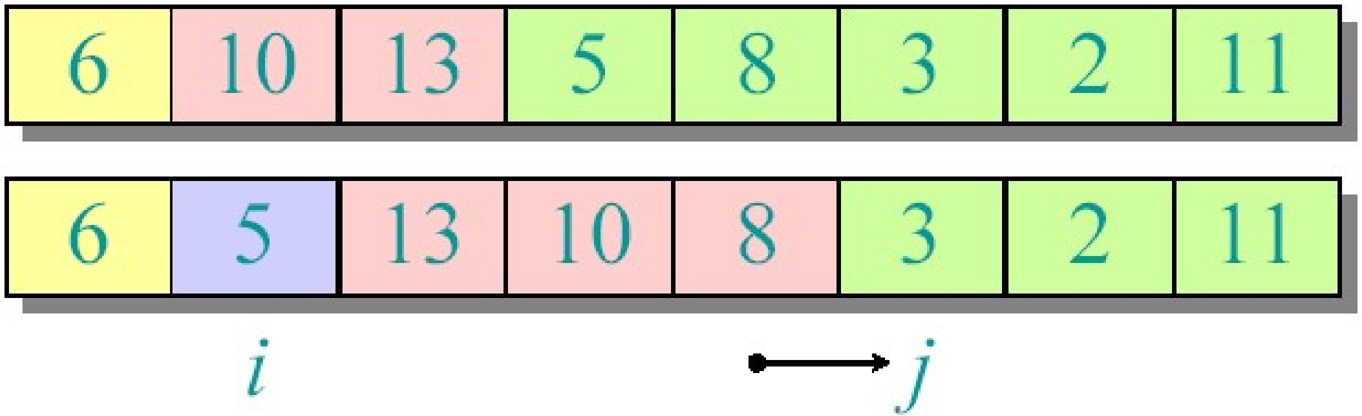

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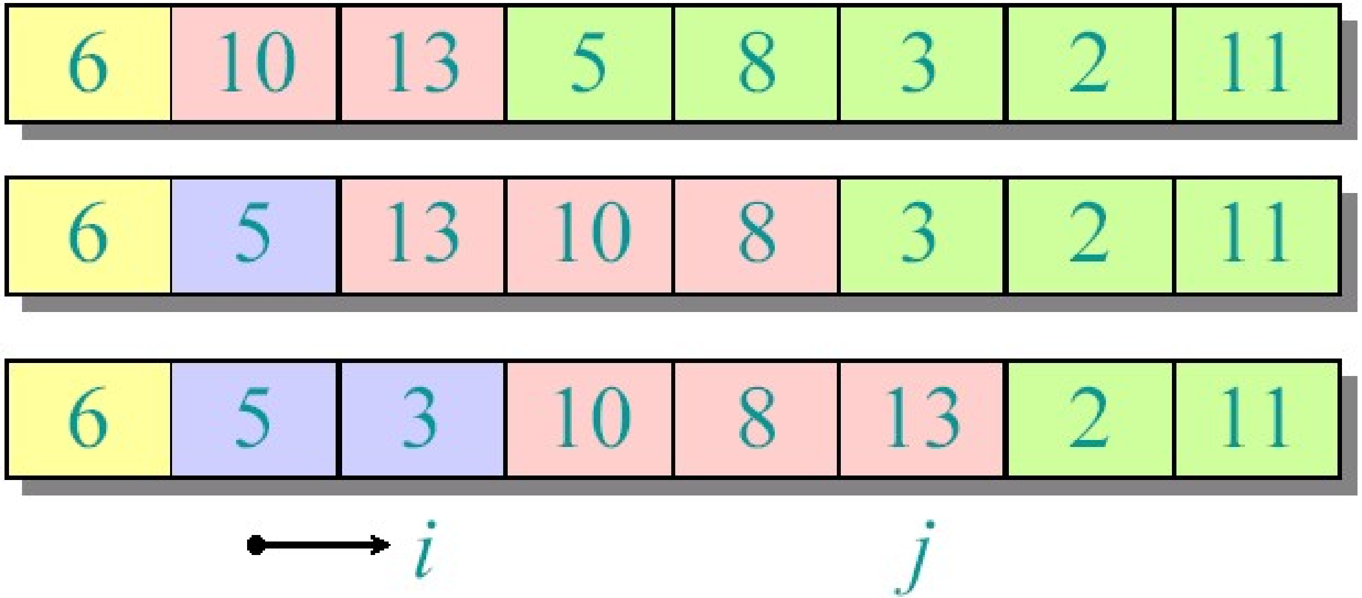

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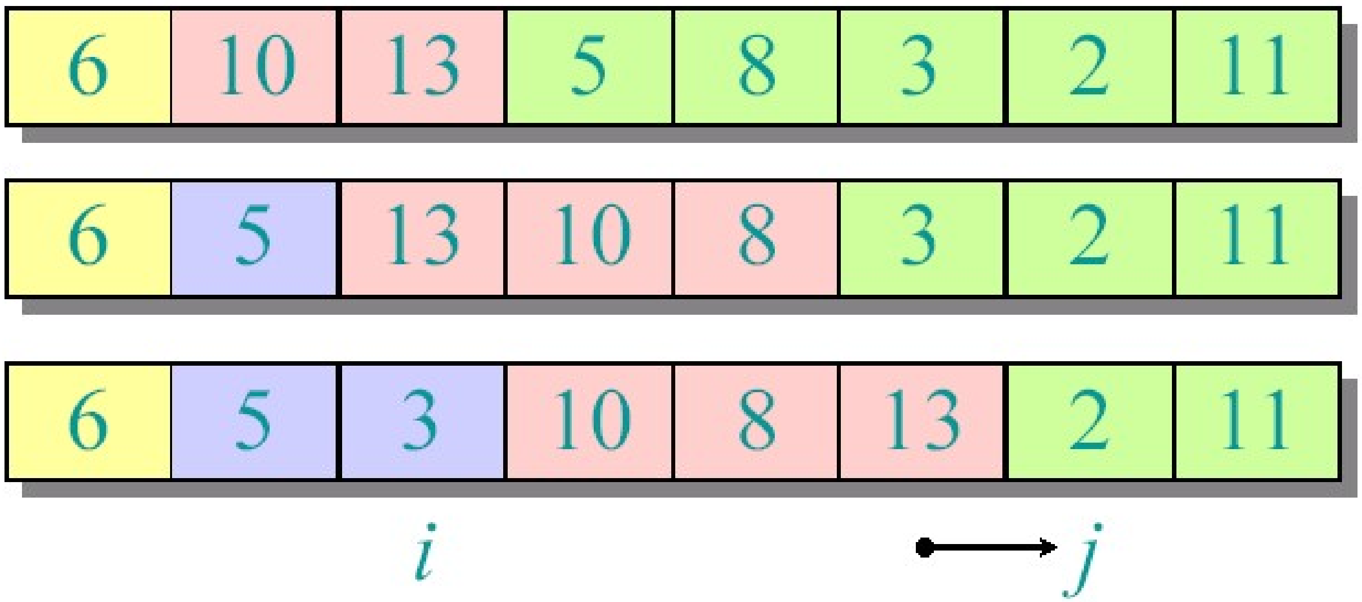

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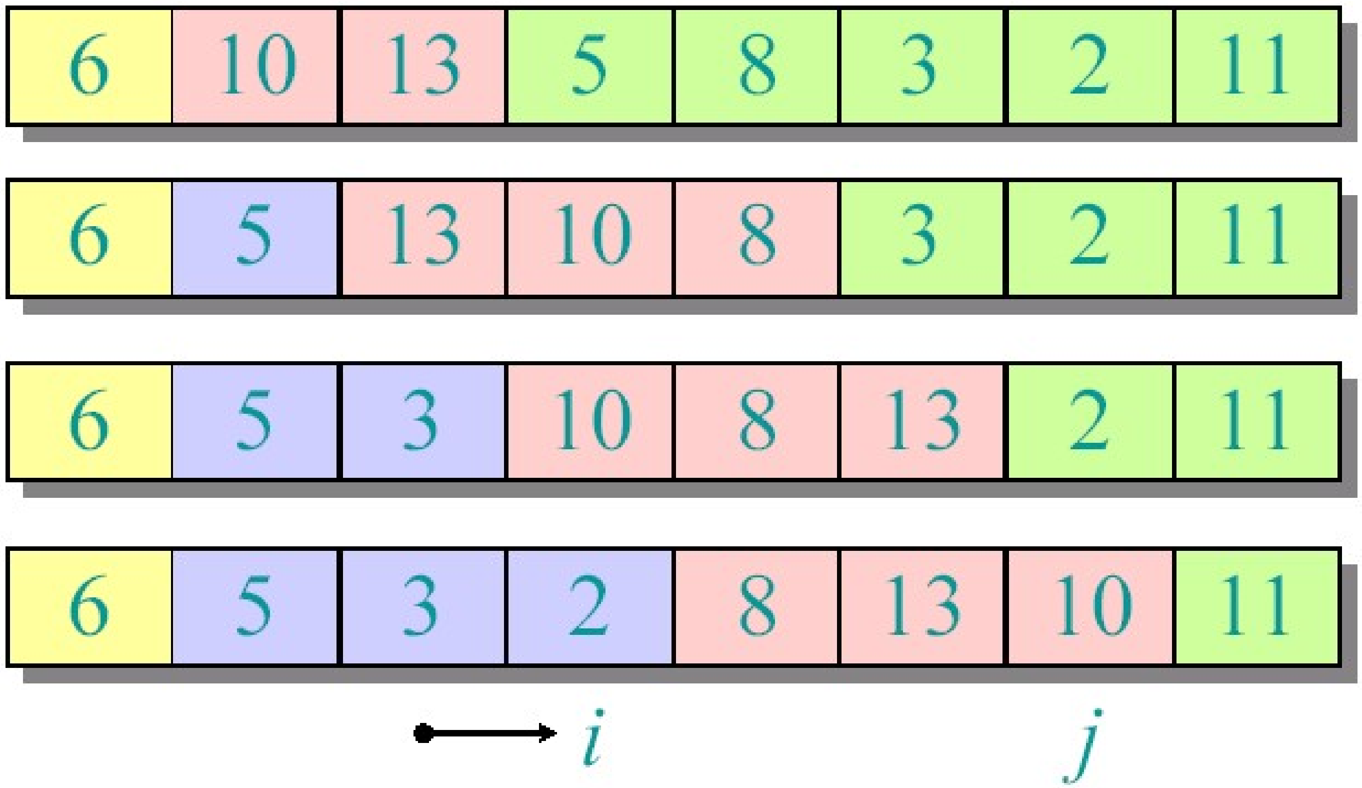

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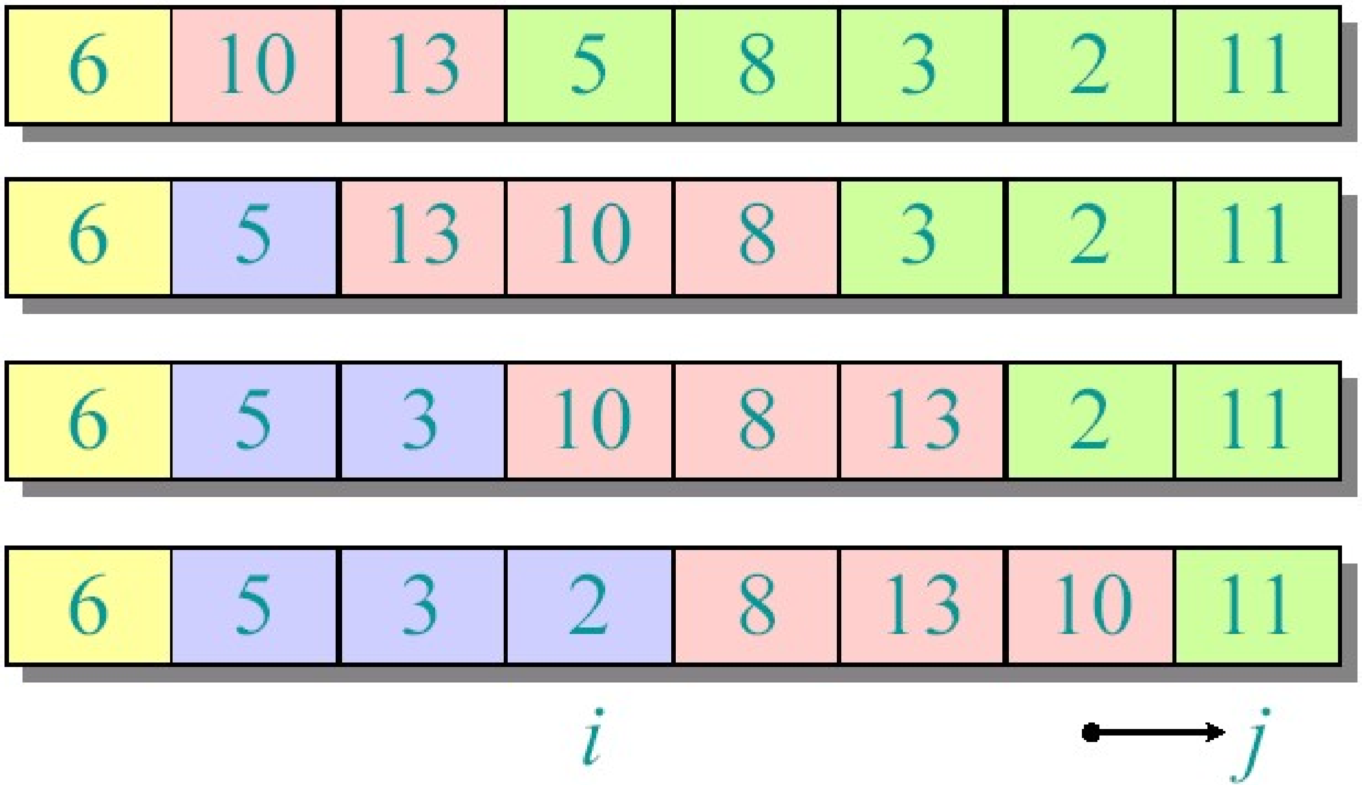

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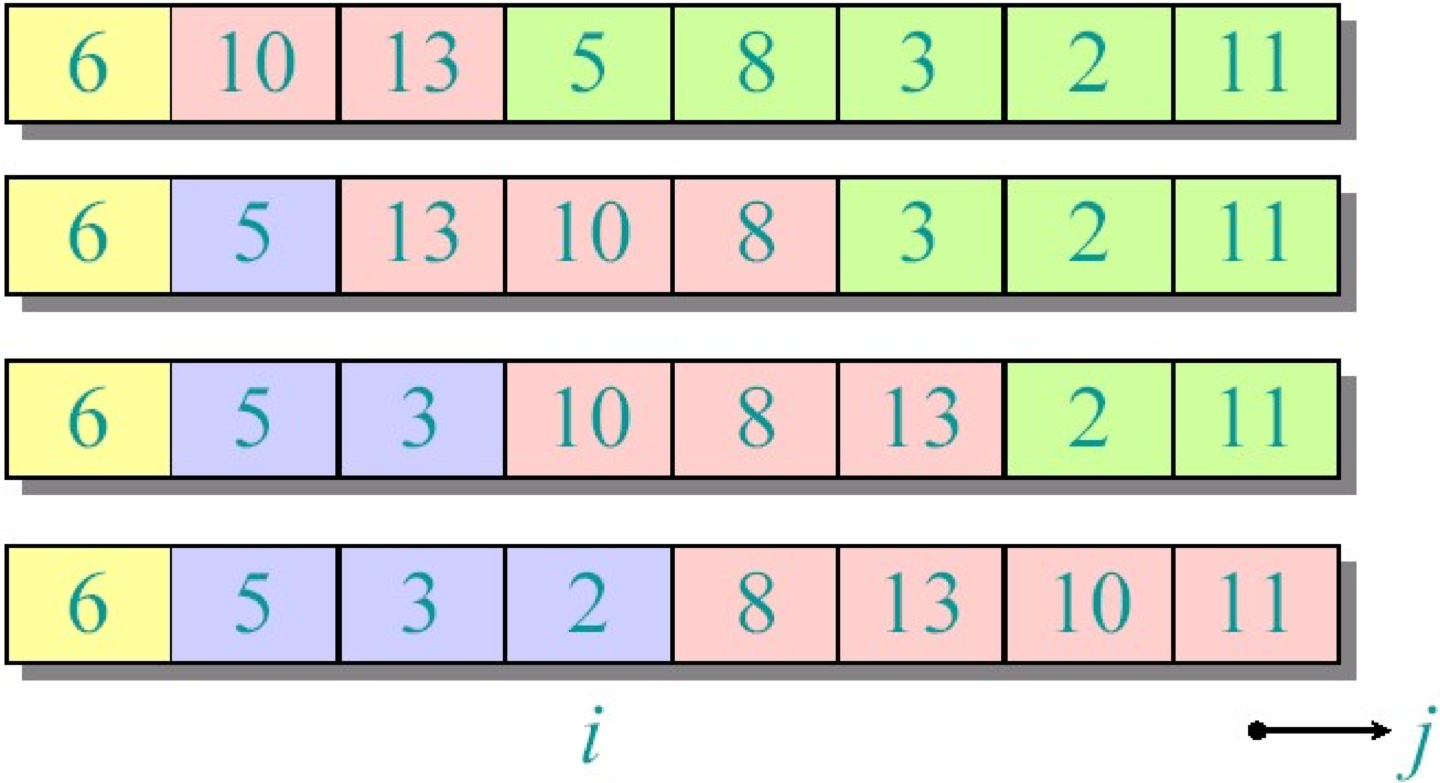
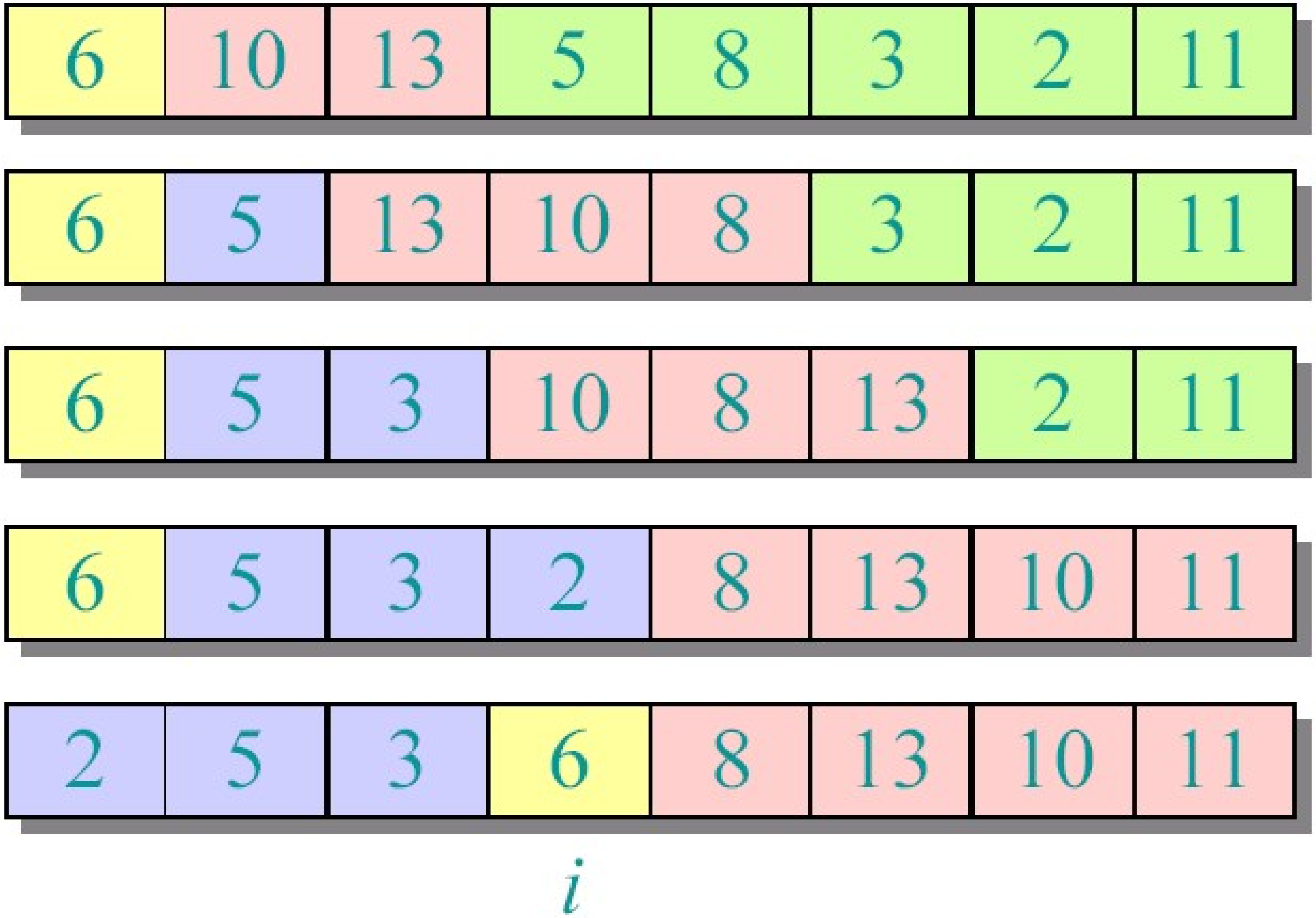


Example of Partitioning



Running Time for PARTITION

The running time of PARTITION on the subarray $A[p...r]$ is $\Theta(n)$ where $n=r-p+1$

Pseudo code for Quicksort

QUICKSORT(A, p, r)
  if $p < r$
    then $q \leftarrow$ PARTITION(A, p, r)
      QUICKSORT(A, p, $q-1$)
      QUICKSORT(A, $q+1$, r)

Initial call: QUICKSORT(A, 1, n)

Analysis of Quicksort

- Assume all input elements are distinct.
- In practice, there are better partitioning algorithms for when duplicate input elements may exist.
- Let $T(n)$ = worst-case running time on an array of n elements.

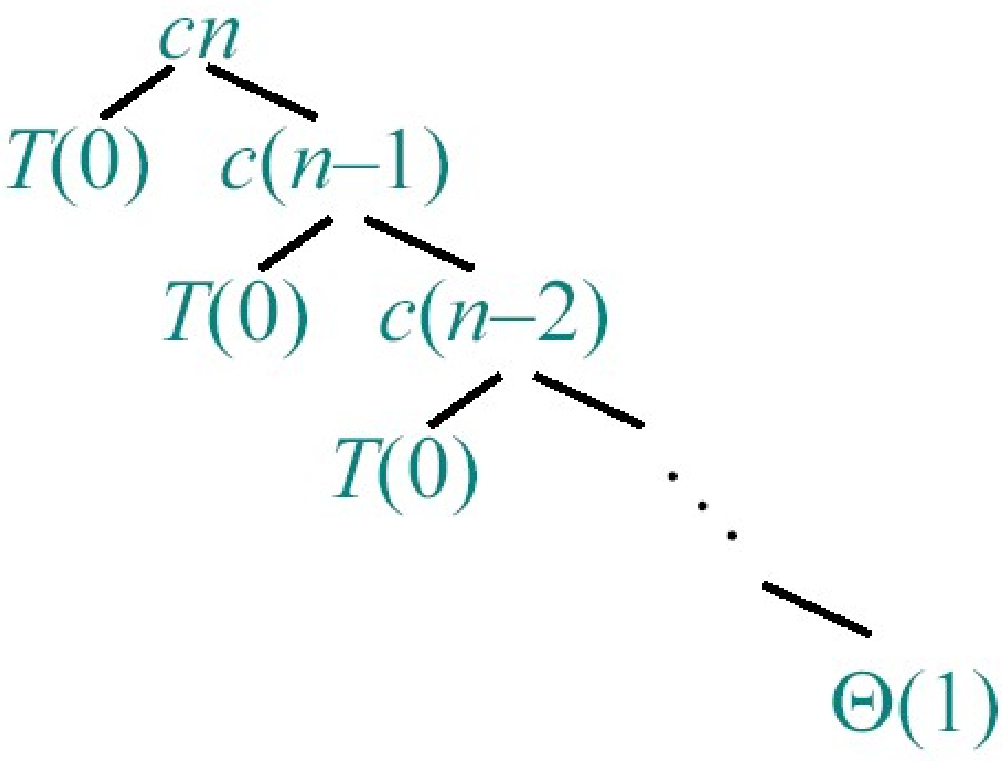
Worst-case of Quicksort

- Input sorted or reverse sorted.
- Partition around min or max element.
- One side of partition always has no elements.

$$
\begin{aligned}
T(n) &= T(0) + T(n-1) + \Theta(n) \\
&= \Theta(1) + T(n-1) + \Theta(n) \\
&= T(n-1) + \Theta(n) \\
&= \Theta(n^2) \qquad \textit{(arithmetic series)}
\end{aligned}
$$

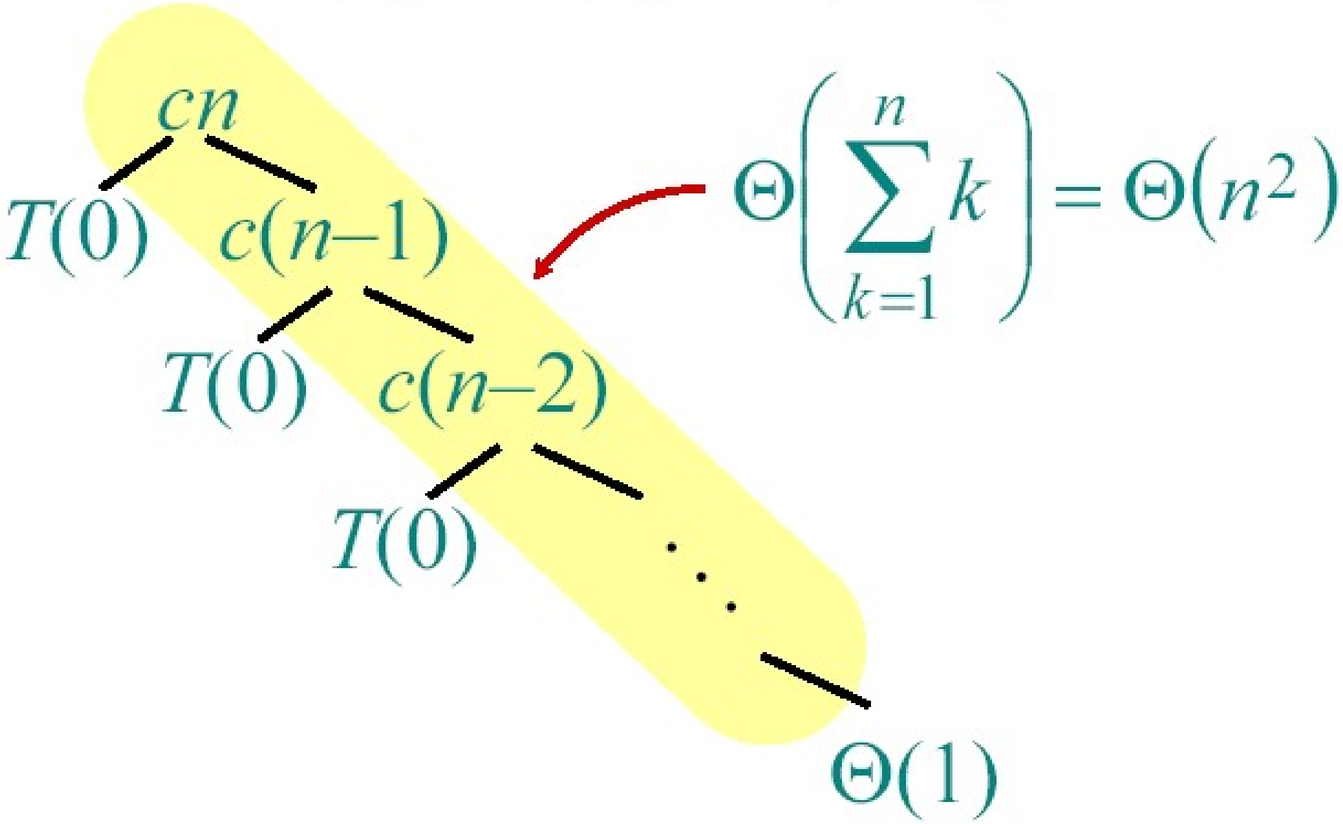
Worst-case Decision Tree

$$T(n) = T(0) + T(n-1) + cn$$



Worst-case Decision Tree

$$T(n) = T(0) + T(n-1) + cn$$



Worst-case Analysis

For the worst case, we can write the recurrence equation as

$$T(n) = \max_{0 \le q \le n-1} \{T(q) + T(n-q-1)\} + \Theta(n)$$

We guess that $T(n) \le cn^2$

$$\begin{aligned} T(n) &\le \max_{0 \le q \le n-1} \{cq^2 + c(n-q-1)^2\} + \Theta(n) \\ &= c \cdot \max_{0 \le q \le n-1} \{q^2 + (n-q-1)^2\} + \Theta(n) \end{aligned}$$

Worst-case Analysis

This expression achieves a maximum at either end points:

***q*=0 or *q*=*n*-1.**

Using the maximum of *T*(*n*) we have

$$T(n) \le \max_{0 \le q \le n-1} \{c\cancel{q}^2 + c(n - \cancel{q} - 1)^2\} + \Theta(n)$$

$$T(n) \le cn^2 - c(2n-1) + \Theta(n)$$
$$\le cn^2$$

Thus

$$T(n) = O(n^2).$$

Worst-case Analysis

We can also show that the recurrence equation as

$$T(n) = \max_{0 \le q \le n-1} \{T(q) + T(n-q-1)\} + \Theta(n)$$

Has a solution of $T(n) = \Omega(n^2)$

We guess that $T(n) \ge cn^2$

$$T(n) \ge \max_{0 \le q \le n-1} \{cq^2 + c(n-q-1)^2\} + \Theta(n)$$

$$= c \cdot \max_{0 \le q \le n-1} \{q^2 + (n-q-1)^2\} + \Theta(n)$$

Worst-case Analysis

Using the maximum of $T(n)$ we have

$$
\begin{aligned}
T(n) &\geq cn^2 - c(2n-1) + \Theta(n) \\
&= cn^2 - c(2n-1) + c_1 n \\
&= cn^2 + (c_1 - 2c)n + c \\
&\geq cn^2 + (c_1 - 2c)n
\end{aligned}
$$

We can pick the constant c_1 large enough so that $c_1 - 2c \geq 0$
and $T(n) \geq cn^2 = \Omega(n^2)$

Thus the worst case running time of quicksort is $\Theta(n^2)$

Best-case Analysis

(For intuition only!)

If we're lucky, PARTITION splits the array evenly:

$$T(n) = 2T(n/2) + \Theta(n)$$
$$= \Theta(n \lg n) \qquad \text{(same as merge sort)}$$

What if the split is always $\frac{1}{10} : \frac{9}{10}$?

$$T(n) = T\left(\tfrac{1}{10}n\right) + T\left(\tfrac{9}{10}n\right) + \Theta(n)$$

What is the solution to this recurrence?

Analysis of Almost Best-case

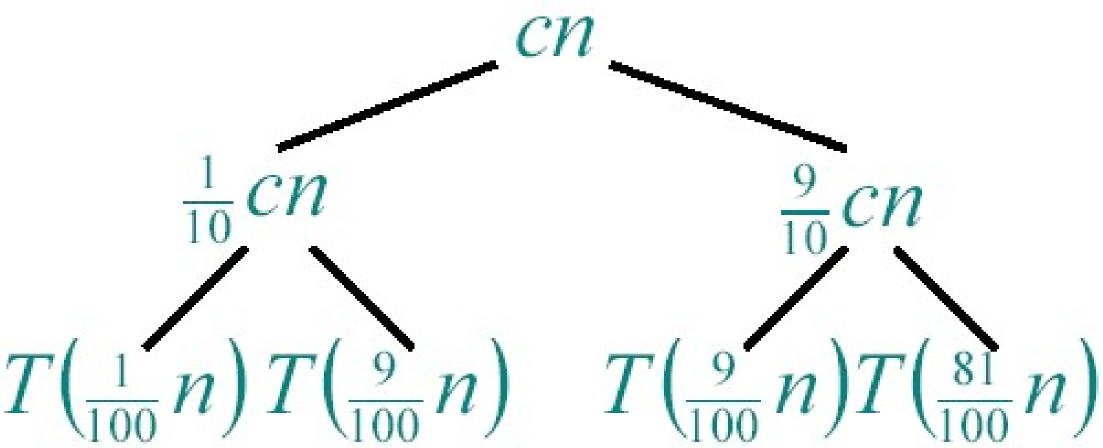
$$T(n)$$

Analysis of Almost Best-case

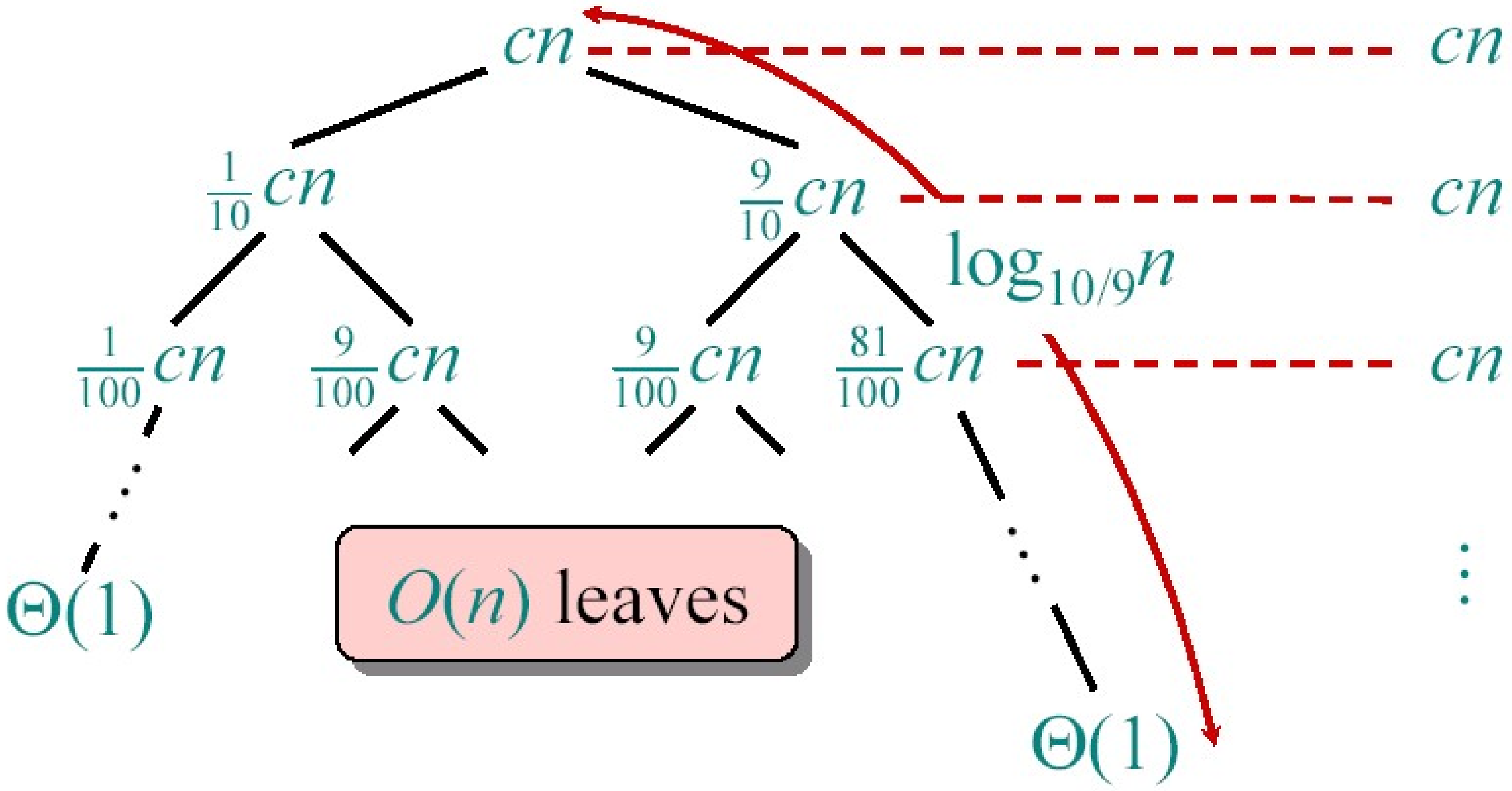
$$cn$$

$$T\left(\frac{1}{10}n\right) \qquad\qquad T\left(\frac{9}{10}n\right)$$

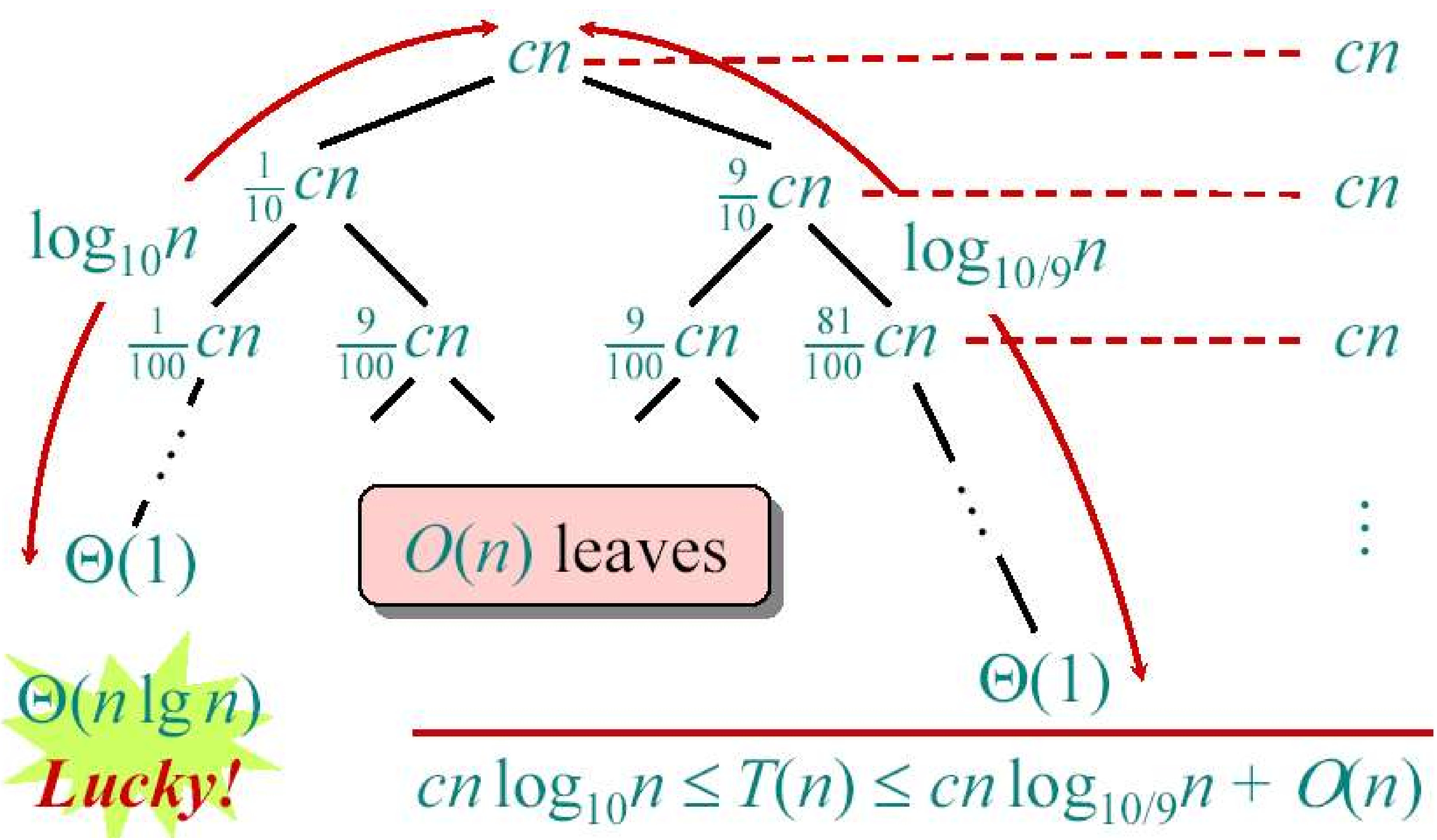
Analysis of Almost Best-case



Analysis of Almost Best-case



Analysis of Almost Best-case



$$cn \log_{10} n \le T(n) \le cn \log_{10/9} n + O(n)$$

Best-case Analysis

For the best case, we can write the recurrence equation as

$$T(n) = \min_{0 \le q \le n-1} \{T(q) + T(n-q-1)\} + \Theta(n)$$

We guess that

$$T(n) \ge cn \log n = \Omega(n \log n)$$

Best-case Analysis

$$T(n) \geq \min_{0 \leq q \leq n-1} \{cq \log q + c(n-q-1)\log(n-q-1)\} + \Theta(n)$$

$$= c \cdot \min_{0 \leq q \leq n-1} \{q \log q + (n-q-1)\log(n-q-1)\} + \Theta(n)$$

Let $Q = q \log q + (n-q-1)\log(n-q-1)$

$$dQ / dq = c\{\frac{q}{q} + \log q - \log(n-q-1) - \frac{(n-q-1)}{(n-q-1)}\} = 0$$

$$\Rightarrow q = \frac{n-1}{2}$$

Best-case Analysis

This expression achieves a minimum at

$$q = \frac{n-1}{2}$$

Using the minimum of *T*(*n*) we have

$$
\begin{aligned}
T(n) &\geq c\{\frac{n-1}{2}\log\frac{n-1}{2} + \frac{n-1}{2}\log\frac{n-1}{2}\} + \Theta(n) \\
&= cn\log(n-1) + c(n-1) + \Theta(n) \\
&= cn\log(n-1) + \Theta(n) \\
&\geq cn\log n \\
&= \Omega(n\log n)
\end{aligned}
$$

More Intuition

Suppose we alternate lucky, unlucky, lucky, unlucky, lucky, ….

$$L(n) = 2U(n/2) + \Theta(n) \quad \textit{lucky}$$

$$U(n) = L(n-1) + \Theta(n) \quad \textit{unlucky}$$

Solving:

$$\begin{aligned} L(n) &= 2(L(n/2-1) + \Theta(n/2)) + \Theta(n) \\ &= 2L(n/2-1) + \Theta(n) \\ &= \Theta(n \lg n) \end{aligned}$$

Lucky!

How can we make sure we are usually lucky?

Randomized Quicksort

IDEA: Partition around a ***random*** element.

- Running time is independent of the input order.
- No assumptions need to be made about the input distribution.
- No specific input elicits the worst-case behavior.
- The worst case is determined only by the output of a random-number generator.

Randomized Quicksort

Standard Problematic Algorithm :

QUICKSORT(A, p, r)
  if $p < r$
    then $q \leftarrow$ PARTITION(A, p, r)
      QUICKSORT($A, p, q-1$)
      QUICKSORT($A, q+1, r$)

Initial call: QUICKSORT(A, 1, n)

Randomized Quicksort

```
RANDOMIZED-PARTITION (A, p, r)
    i←RANDOM(p, r)
    exchange A[r]↔A[i]
    return PARTITION(A, p, r)
```

```
RANDOMIZED-QUICKSORT (A,p,r)
    if p<r
    then q←RANDOMIZED-PARTITION (A, p, r)
            RANDOMIZED-QUICKSORT (A, p, q-1)
            RANDOMIZED-QUICKSORT (A, q+1, r)
```

Randomized Quicksort

Let $T(n)$ = the random variable for the running time of randomized quicksort on an input of size n, assuming random numbers are independent.

For $k = 0, 1, \ldots, n-1$, define the ***indicator random variable***

$$X_k = \begin{cases} 1 & \text{if PARTITION generates a } k : n-k-1 \text{ split,} \\ 0 & \text{otherwise.} \end{cases}$$

$E[X_k] = \Pr\{X_k = 1\} = 1/n$, since all splits are equally likely, assuming elements are distinct.

Randomized Quicksort Analysis

$$T(n) = \begin{cases} T(0) + T(n-1) + \Theta(n) & \text{if } 0 : n-1 \text{ split,} \\ T(1) + T(n-2) + \Theta(n) & \text{if } 1 : n-2 \text{ split,} \\ \quad\vdots & \\ T(n-1) + T(0) + \Theta(n) & \text{if } n-1 : 0 \text{ split,} \end{cases}$$

$$= \sum_{k=0}^{n-1} X_k \left(T(k) + T(n-k-1) + \Theta(n) \right).$$

Calculating Expectation

$$E[T(n)] = E\left[\sum_{k=0}^{n-1} X_k\left(T(k) + T(n-k-1) + \Theta(n)\right)\right]$$

Take expectation in both side

$$= \sum_{k=0}^{n-1} E\left[X_k\left(T(k) + T(n-k-1) + \Theta(n)\right)\right]$$

$$= \sum_{k=0}^{n-1} E[X_k] \cdot E[T(k) + T(n-k-1) + \Theta(n)]$$

Independence of X_k from other random choice

$$= \frac{1}{n}\sum_{k=0}^{n-1} E[T(k)] + \frac{1}{n}\sum_{k=0}^{n-1} E[T(n-k-1)] + \frac{1}{n}\sum_{k=0}^{n-1} \Theta(n)$$

Linearity of expectation; $E[X_k] = 1/n$.

Calculating Expectation

$$E[T(n)] = E\left[\sum_{k=0}^{n-1} X_k\left(T(k)+T(n-k-1)+\Theta(n)\right)\right]$$

$$= \sum_{k=0}^{n-1} E\left[X_k\left(T(k)+T(n-k-1)+\Theta(n)\right)\right]$$

$$= \sum_{k=0}^{n-1} E[X_k]\cdot E[T(k)+T(n-k-1)+\Theta(n)]$$

$$= \frac{1}{n}\sum_{k=0}^{n-1} E[T(k)] + \frac{1}{n}\sum_{k=0}^{n-1} E[T(n-k-1)] + \frac{1}{n}\sum_{k=0}^{n-1}\Theta(n)$$

$$= \frac{2}{n}\sum_{k=1}^{n-1} E[T(k)] + \Theta(n)$$

Summations have identical terms.

Calculating Expectation

$$E[T(n)] = \frac{2}{n}\sum_{k=2}^{n-1} E[T(k)] + \Theta(n)$$

(The $k = 0, 1$ terms can be absorbed in the $\Theta(n)$.)

Prove: $E[T(n)] \le a n \lg n$ for constant $a > 0$.

- Choose a large enough so that $a n \lg n$ dominates $E[T(n)]$ for sufficiently small $n \ge 2$.

Use fact: $\sum_{k=2}^{n-1} k \lg k \le \frac{1}{2} n^2 \lg n - \frac{1}{8} n^2$ (exercise).

Calculating Expectation

$$\sum_{k=1}^{n-1} k \lg k = \sum_{k=1}^{\lceil n/2 \rceil - 1} k \lg k + \sum_{k=\lceil n/2 \rceil}^{n-1} k \lg k$$

The $\lg k$ in the first summation is bounded above by $\lg (n/2) = \lg n - 1$

The $\lg k$ in the second summation is bounded above by $\lg n$

$$\le (\lg n - 1) \sum_{k=1}^{\lceil n/2 \rceil - 1} k + \lg n \sum_{k=\lceil n/2 \rceil}^{n-1} k$$

$$= \lg n \sum_{k=1}^{n-1} k - \sum_{k=1}^{\lceil n/2 \rceil - 1} k$$

$$\le \frac{1}{2} n(n-1) \lg n - \frac{1}{2} (\frac{n}{2} - 1) \frac{n}{2}$$

$$\le \frac{1}{2} n^2 \lg n - \frac{1}{8} n^2$$

For n ≥ 2 this is the bound.

Substitution Method

$$E[T(n)] \le \frac{2}{n}\sum_{k=2}^{n-1} ak \lg k + \Theta(n)$$

$$= \frac{2a}{n}\left(\frac{1}{2}n^2 \lg n - \frac{1}{8}n^2\right) + \Theta(n)$$

$$= an\lg n - \left(\frac{an}{4} - \Theta(n)\right)$$

Desired - Residual

$$\le an\lg n,$$

if a is chosen large enough so that $an/4$ dominates the $\Theta(n)$.

Quicksort: Summary

- Quicksort is a great general-purpose sorting algorithm.
- Quicksort is typically over twice as fast as merge sort.
- Quicksort can benefit substantially from ***code tuning***.
- Quicksort behaves well even with caching and virtual memory.