CSE 2315 - Discrete Structures  
Lecture 12: Relations - Fall 2010

1 Motivation

- Given different sets it is important to be able to manipulate different elements and to be able to relate them to each other.
- Operations present transformations from elements of a set to other elements in the set.
- But: Often we are interested in properties like “is less than”, “is a divisor of”, etc., which are not operations.
- Relations between different objects can be important for applications like database retrieval or scheduling.

2 Relations

- Relations establish a connection between various elements of sets.
  - Examples:
    * $x < y$ is a relation for all sets of numbers.
    * $x \subseteq y$ is a relations for power sets $\varnothing$.
    * $x$ is the father of $y$ is a relation on a set of people.

- A general binary relation $\rho$ on a set $S$ can be characterized by the set of all ordered pairs of elements which have the given correlation.

$$x \rho y \leftrightarrow (x, y) \in \rho$$
$$\rho \subseteq S \times S.$$  

- Description of a relation as a set allows to define relations which do not have an apparent verbal description.
  - Examples:
    * Given the set $S = \{1, 2, 3\}$, the relation $x \rho y \leftrightarrow x < y$ can be represented by $\rho = \{(1, 2), (1, 3), (2, 3)\}$
    * $\rho = \{(1, 1), (3, 3)\}$ on the set $S = \{1, 2, 3\}$ represents the relation: $x = y$ and $x$ is odd.
Given the set $S = \{1, 2, 3, 4, 5\}$, the relation $x^2 < y$ can be represented by
\{(1, 2), (1, 3), (1, 4), (1, 5), (2, 5)\}.

Note: For infinite sets it is generally not possible to specify a binary relation by the set of all ordered pairs. Instead most binary relations are specified by a binary predicate representing the characterizing property.

**• Binary relations can also define a connection between elements of different sets. $\rho \subseteq S \times T$**

**Examples:**

* Given the sets $S = \{Ford, Honda, GM, Toyota\}$ and $T = \{USA, Japan\}$, the relation $\rho = \{(Ford, America), (Honda, Japan), (GM, America), (Toyota, Japan)\}$ represents “company $x$ is located in country $y$”.

$S = \mathbb{N}$, $T = \mathbb{R}$, $x\rho y \iff y = \sqrt{x}$

**• Interpreting binary relations as sets of ordered pairs allows to combine multiple relations using set operations. This allows to generate more complex relations.**

- $x(\rho \cup \sigma)y \iff x\rho y$ or $x\sigma y$
- $x(\rho \cap \sigma)y \iff x\rho y$ and $x\sigma y$
- $x\bar{\rho}y \iff \neg x\rho y$

**• n-ary relations can be defined as subsets of $S_1 \times S_2 \times \ldots \times S_n$ and can be represented by corresponding n-ary predicates of the characteristic properties.**

### 3 Properties of Relations

**• Binary relations can associate two elements in various ways.**

- Relations are one-to-one if every element in the first set relates to at most one element in the second set and vice versa.
- Relations are one-to-many if every element in the second set relates to at most one element in the first set but at least one element in the first set relates to more than one element in the second set.
- Relations are many-to-one if every element in the first set relates to at most one element in the second set but at least one element in the second set is related to more than one element of the first set.
- Relations are many-to-many if at least one element in each of the sets is related to more than one element of the other set.

**• Different types of binary relations on a set $S$ allow different techniques to be applied.**

- $x \leq y$, for example imposes an ordering between all the elements of $\mathbb{N}$.
- $x = y$ divides the set $\mathbb{N}$ into separate items since every element only relates to itself.
• To differentiate some important types of binary relations a set of properties can be defined.

  - A binary relation \( \rho \) is reflexive if for every element \( x \in S \), \((x, x) \in \rho\).
    \[ \forall x \in S \rightarrow (x, x) \in \rho \]
    * Examples:
      \( \rho x y \Leftrightarrow x = y \) on the set \( S = \mathbb{N} \) is reflexive
      \( \rho x y \Leftrightarrow x \leq y \) on the set \( S = \mathbb{N} \) is reflexive
      \( \rho x y \Leftrightarrow x < y \) on the set \( S = \mathbb{N} \) is not reflexive

  - A binary relation \( \rho \) is symmetric if for any element \((x, y) \in \rho\) also \((y, x) \in \rho\).
    \[ (\forall x)(\forall y)(x \in S \land y \in S \land (x, y) \in \rho \rightarrow (y, x) \in \rho) \]
    * Examples: \( \rho x y \Leftrightarrow x = y \) on the set \( S = \mathbb{N} \) is symmetric
      \( \rho x y \Leftrightarrow x \leq y \) on the set \( S = \mathbb{N} \) is not symmetric
      \( \rho x y \Leftrightarrow x < y \) on the set \( S = \mathbb{N} \) is not symmetric
    * Note: antisymmetric is not the opposite of symmetric. \( x = y \), for example, is symmetric and antisymmetric.

  - A binary relation \( \rho \) is antisymmetric if whenever \((x, y) \) and \((y, x) \) are elements of \( \rho \), then \( x = y \).
    \[ (\forall x)(\forall y)(x \in S \land y \in S \land (x, y) \in \rho \land (y, x) \in \rho \rightarrow x = y) \]
    * Examples: \( \rho x y \Leftrightarrow x = y \) on the set \( S = \mathbb{N} \) is antisymmetric
      \( \rho x y \Leftrightarrow x \leq y \) on the set \( S = \mathbb{N} \) is antisymmetric
      \( \rho x y \Leftrightarrow x < y \) on the set \( S = \mathbb{N} \) is antisymmetric
    * Note: antisymmetric is not the opposite of symmetric. \( x = y \), for example, is symmetric and antisymmetric.

  - A binary relation \( \rho \) is transitive if whenever \((x, y) \) and \((y, z) \) are in \( \rho \), then also \((x, z) \) is in \( \rho \).
    \[ (\forall x)(\forall y)(\forall z)(x \in S \land y \in S \land z \in S \land (x, y) \in \rho \land (y, z) \in \rho \rightarrow (x, z) \in \rho) \]
    * Examples: \( \rho x y \Leftrightarrow x = y \) on the set \( S = \mathbb{N} \) is transitive
      \( \rho x y \Leftrightarrow x \leq y \) on the set \( S = \mathbb{N} \) is transitive
      \( \rho x y \Leftrightarrow x < y \) on the set \( S = \mathbb{N} \) is transitive \( x \) is a divisor of \( y \) on the set \( S = \mathbb{N} \) is not transitive

• In certain situations it might be useful to extend a relations such that it has one of the above properties. The smallest superset \( \rho^* \) of \( \rho \) that has the desired property is called the closure of \( \rho \) on \( S \) with respect to that property.

  - Examples:
    * Given the set \( S = \{1, 2, 3\} \), the reflexive closure of \( \rho = \{(1, 1), (1, 2), (2, 3), (3, 3)\} \) is \( \rho^* = \rho \cup \{(2, 2)\} \).
    * Given the set \( S = \{1, 2, 3\} \), the transitive closure of \( \rho = \{(1, 1), (1, 2), (2, 3), (3, 3)\} \) is \( \rho^* = \rho \cup \{(1, 3)\} \).
    * Given the set \( S = \{1, 2, 3\} \), the reflexive, symmetric and transitive closure of \( \rho = \{(1, 1), (1, 2), (2, 3), (3, 3)\} \) is \( \rho^* = \rho \cup \{(2, 2), (2, 1), (3, 2), (1, 3), (3, 1)\} \).
    * Note: If a relation is not antisymmetric, then no antisymmetric closure exists.
4 Binary Relation Types

- Different properties allow different techniques to be applied to a relation.

- A partial ordering is a binary relation $\rho$ that is reflexive, antisymmetric, and transitive.
  - In a partial ordering on $S$ ($S, \rho$) is called a partially ordered set (poset). A general partial
    ordering is often indicated by $\leq$.
  - Given a partial ordering $\leq$, if $x \leq y$ and $x \neq y$ then $x \prec y$. In this case $x$ is called the
    predecessor of $y$ and $y$ is the successor of $x$.
  - If $x \prec y$ and there is no $z$ such that $x \prec z$ and $z \prec y$, then $x$ is the immediate predecessor
    of $y$.
  - A relation with these properties is called a partial ordering because we can write all ele-
    ments in $S$ in a sequence $s_1, s_2, ..., s_n$ such that if $(s_i, s_k) \in \leq$ then $i \leq j$.
  - Note: As opposed to a total ordering it is not required that for all $i < j$ $(s_i, s_j) \in \leq$.
  - Examples:
    - $\subseteq$ is a partial ordering on an arbitrary power set.
    - $\leq$ is a partial ordering on $\mathbb{N}$.
    - $=$ is a partial ordering on $\mathbb{N}$.
    - $<$ is not a partial ordering on $\mathbb{N}$ (it is not reflexive).
  - In a partial ordering the least element $x$ is an element such that for all elements $y$ in $S$, $x \leq y$.
  - A minimal element $x$ is an element such that there is no element $y$ in $S$ with $y < x$.
  - Similarly the greatest element and the maximal element are defined.
  - Note: There does not have to be a least or a greatest element for a relation $\leq$.
  - A partial ordering can be visualized using Hasse diagrams where each element in $S$ is
    represented by a node (or vertex) and the immediate predecessor of a node is above the
    node, connected with a straight-line edge.
  - Examples:
    - The equality relation on the set $S = \{1, 2, 3\}$
    - The $\leq$ relation on the set $S = \{1, 2, 3\}$.
    - The $\subseteq$ relation on $\varnothing(\{1, 2, 3\})$.
  - Partial orderings can represent the precedence constraints in a program or manufacturing
    application.
  - Example: Task $A$ (1 hr) has to be performed before Task $B$ (2 hr). Task $C$ (1.5 hr)
    requires task $A$ to be finished. Task $D$ (1 hr) has to be performed after task $B$. Task $E$ (0.5 hr)
    requires the completion of $D$ and $C$. 
For the evaluation of such a schedule Hasse diagrams are often slightly modified into PERT (program evaluation and review technique) charts. The main difference is that the chart is here drawn from left to right.

To construct a total ordering (i.e. complete sequential schedule) topological sorting is used.

- Topological sorting picks a minimal element, removes it from the set and adds it to the total ordering. This is repeated until no elements are left.
- There can be multiple total orderings from one partial ordering.

An equivalence relation on a set $S$ is a binary relation that is reflexive, symmetric, and transitive.

- An equivalence relation $\rho$ on a set $S$ divides the set into separate partitions called blocks. Each of these blocks or equivalence classes can be characterized by a characteristic element $x$ and written as $[x]$.

$$[x] = \{ y | y \in S \land x \rho y \}$$

- Every partition of a set $S$ represents an equivalence relation.

- Examples:
  - $x = y$ for the set $\mathbb{N}$ is an equivalence relation with equivalence classes $[0], [1], \ldots$.
  - $x = y$ is an equivalence relation for the set $\mathbb{Q}$. Equivalence classes here consist of all equivalent fractions, e.g. $\frac{1}{2} \sim \frac{2}{4} \in \left[\frac{1}{2}\right]$.
  - Congruence modulo $n$ is an equivalence relation on $\mathbb{N}$. $x \equiv_n y \iff x = y + z \cdot n$, $z \in \mathbb{Z}$. Equivalence classes here are $[0], [1], \ldots, [n - 1]$.
  - $x + y$ is even is an equivalence relation for $\mathbb{N}^+$ ($\{ x \in \mathbb{N} \land x > 0 \}$) which partitions the space into all odd and all even numbers. The equivalence classes are $[1]$ and $[2]$. 