1 Motivation

- Propositional logic provides a formalism to translate sentences into formal symbols.
  - Can analyze sentences without the “clutter” introduced by the “meaning” and ambiguity of natural language.
  - Truth values are absolute. There is no “maybe” or “probably”.
  - A set of well defined operators allows to express interrelations among statements without ambiguity.
  - In formal logic there is only one correct answer. Everybody has to reach the same conclusion.

- Truth tables allow to determine the truth values of compound statements.

- Tautologies as an important special case are statements which are true under all possible conditions (we will come back to these).

- Often we want to determine if a given conclusion can be drawn from a set of facts or, in other words, if an argument someone made is valid. So far, however, we have only determined the truth values of a single statement.

- An argument generally involves a number of interrelated facts from which a conclusion is drawn. They can become very long and in natural language it is often hard to follow a long argument without getting confused. It can therefore be hard to assess if such an argument is valid. In formal logic there is no ambiguity and interpretation.
  - Translating the argument into formal logic disconnects its structure from the “meaning” and makes testing easier.

2 Logical Arguments

- An argument in propositional logic takes the following form:
  \[ P_1 \land P_2 \land \ldots \land P_n \rightarrow Q \]
  where \( \{P_i\} \) is the set of hypotheses and \( Q \) is the conclusion.
Lecture 3: Propositional Calculus

Note: \( P_i \) and \( Q \) are general, compound statements and can thus take forms such as \((A \rightarrow B) \rightarrow C \) or \( A \lor \neg B \).

- An argument in formal logic is valid if and only if it is a tautology.
  - Arguments are valid solely based on their structure. The “meaning” of the original sentences is irrelevant.

- The validity of an argument can be tested in different ways:
  - Tautology test using truth tables
    - Simple and straightforward to do.
    - Table becomes very big for larger arguments.
    - There are situations in formal logic where you can not write down truth tables. (We will see this later)
  - Proof sequences using propositional calculus

3 Proofs using Propositional Calculus

- In propositional calculus rules are used to transform statements.
- Transformations within a statement:
  - In mathematics we would prove the correctness of an equation by changing the structure on one side using a set of operations until it matches the other:
    \[
    2 \ast (x + 1) \ast \frac{(x - 1)}{3} \iff \frac{2}{3} \ast (x^2 - 1)
    \]
    \[
    2 \ast (x + 1) \ast \frac{(x - 1)}{3} = \frac{2}{3} \ast (x + 1) \ast (x - 1) \\
    = \frac{2}{3} \ast (x^2 - 1)
    \]
    Legal operations in mathematics have to preserve the correctness of the formula (for equations this implies they have to preserve the value).
  - In formal logic we can do the same thing. Whenever the structure of a statement is transformed, its truth value has to be preserved. Tautological equivalences contain two statements which have different structure but identical truth values.
\[(A \land B) \land B \Leftrightarrow (A \land B)\]

\[(A \land B) \land B \Leftrightarrow (A \lor B) \land B \quad \text{(De Morgan)}\]

\[ \Leftrightarrow (A \land B) \lor (\overline{B} \land B) \quad \text{(Distributivity)}\]

\[ \Leftrightarrow (A \land B) \lor F \quad \text{(Complement)}\]

\[ \Leftrightarrow (A \land B) \quad \text{(Identity)}\]

* Equivalence rules:

<table>
<thead>
<tr>
<th>Rule Name</th>
<th>Expression</th>
<th>Equivalent Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>Commutativity</td>
<td>(P \lor Q)</td>
<td>(Q \lor P)</td>
</tr>
<tr>
<td></td>
<td>(P \land Q)</td>
<td>(Q \land P)</td>
</tr>
<tr>
<td>Associativity</td>
<td>((P \lor Q) \lor R)</td>
<td>(P \lor (Q \lor R))</td>
</tr>
<tr>
<td></td>
<td>((P \land Q) \land R)</td>
<td>(P \land (Q \land R))</td>
</tr>
<tr>
<td>De Morgan’s Laws</td>
<td>((P \lor Q))</td>
<td>(\overline{P} \land \overline{Q})</td>
</tr>
<tr>
<td></td>
<td>((P \land Q))</td>
<td>(\overline{P} \lor \overline{Q})</td>
</tr>
<tr>
<td>Implication</td>
<td>(P \rightarrow Q)</td>
<td>(\overline{P} \lor Q)</td>
</tr>
<tr>
<td>Double negation</td>
<td>((\overline{P}))</td>
<td>(P)</td>
</tr>
</tbody>
</table>

Note: Equivalence rules are always invertible.

- In logical arguments we have a set of hypotheses which together lead to the conclusion. Multiple statements often have to be combined to prove the validity.

- Transformations among multiple statements:
  - In mathematics systems of equations are solved by deriving new, legal equations from the already known ones:

    \[ 2 \ast (x + y) = 4 , \ y = 2 \] permits to derive \[ 2 \ast (x + 2) = 4 \]
    \[ 2 \ast (x + y) = 4 , \ y < 2 \] permits to derive \[ 2 \ast (x + 2) > 4 \]

    Legal derivations have to ensure the correctness of the new formula. (Note: Derivations are very often not invertible)

    - In formal logic such derivation rules have to preserve the correct truth values.
Inference rules:

<table>
<thead>
<tr>
<th>Rule Name</th>
<th>From</th>
<th>Can Derive</th>
</tr>
</thead>
<tbody>
<tr>
<td>Conjunction</td>
<td>$P, Q$</td>
<td>$P \land Q$</td>
</tr>
<tr>
<td>Simplification</td>
<td>$P \land Q$</td>
<td>$P, Q$</td>
</tr>
<tr>
<td>Modus ponens</td>
<td>$P, P \rightarrow Q$</td>
<td>$Q$</td>
</tr>
<tr>
<td>Modus tollens</td>
<td>$P \rightarrow Q, \overline{Q}$</td>
<td>$\overline{P}$</td>
</tr>
</tbody>
</table>

Note: Inference rules are generally not invertible.

- To be able to test the validity of general arguments the set of rules has to be complete (it has to provide the means to prove all valid arguments) and correct (it can not result in a valid proof for an invalid argument). The given set of rules is complete and correct for propositional logic.

4 Proof Sequence

- A proof sequence is a sequence of well formed formulas (wffs) which starting from the hypotheses of an argument derived the conclusion by applying equivalence rules and inference rules.

\[
(A \land \overline{B}) \land \overline{B} \rightarrow \overline{A}
\]

1. $(A \land \overline{B})$ hyp
2. $\overline{B}$ hyp
3. $\overline{A} \lor \overline{(\overline{B})}$ 1 De Morgan
4. $\overline{A} \lor B$ 3 dn
5. $A \rightarrow B$ 4 imp
6. $\overline{A}$ 2, 5 mt

- Hint: A proof sequence is not a strict sequence. You don’t have to use the statements at the bottom to derive new ones but rather all statements in the list can be used at any given time. Also statements can be used multiple times.

- Hint: As a rule of thumb: statements of the form $(P \lor Q), (P \land Q)$, or $P \lor Q$ are rarely useful. Convert them using De Morgan and Double negation, and Implication.

- Additional rules are not necessary but can potentially shorten the proof sequences. Note: Additional rules have to be proven first.
– Additional equivalence rules:
  * E.g. Deduction
    \[ P \rightarrow (Q \rightarrow R) \iff P \land Q \rightarrow R \]
    Proof:
    \[
    P \rightarrow (Q \rightarrow R) \iff \overline{P} \lor (Q \rightarrow R) \quad \text{imp} \\
    \iff \overline{P} \lor (Q \lor R) \quad \text{imp} \\
    \iff (\overline{P} \lor Q) \lor R \quad \text{ass} \\
    \iff (P \land Q) \lor R \quad \text{DeMorgan} \\
    \iff P \land Q \rightarrow R \quad \text{imp}
    \]

– Additional inference rules:
  * E.g. Hypothetical Syllogism
    \[(P \rightarrow Q), (Q \rightarrow R) \text{ permit to derive } (P \rightarrow R), \text{ or in other terms } (P \rightarrow Q) \land (Q \rightarrow R) \rightarrow (P \rightarrow R) \text{ is a valid argument}\]
    Proof:
    \[
    (P \rightarrow Q) \land (Q \rightarrow R) \rightarrow (P \rightarrow R) \\
    (P \rightarrow Q) \land (Q \rightarrow R) \land P \rightarrow R \quad \text{Deduction}
    \]
    1. \( (P \rightarrow Q) \) \hyp
    2. \( (Q \rightarrow R) \) \hyp
    3. \( P \) \hyp
    4. \( Q \) \(1, 3 \text{ mp}\)
    5. \( R \) \(2, 4 \text{ mp}\)

• Using proof sequences you can now translate confusing arguments from English into formal logic and then test their validity. That you do not have to pay attention to the “meaning” is here a big advantage.

5 Exercises
• Prove the following:
  - \( A \land (B \rightarrow C) \land ((A \land B) \rightarrow (D \lor \overline{C})) \land B \rightarrow D \)
  - \( ((A \lor \overline{B}) \rightarrow C) \land (C \rightarrow D) \land A \rightarrow D \)
  - \( (\overline{A} \lor B) \land (B \rightarrow C) \rightarrow (A \rightarrow C) \)
– \((A \rightarrow B) \land (\overline{C} \lor A) \land C \rightarrow B\)

– Only if Debbie was in Houston could she have stolen Paul’s wallet yesterday but Paul either lost his wallet or it was stolen. If it was not stolen yesterday or at night, then it was not stolen at all. Also, if someone took the wallet it must have been Debbie. But Debbie was not in Houston and she could not have stolen the wallet at night. Therefore Paul lost his wallet.