1 Motivation

- Formal logic allows to perform detailed, direct proofs of the validity of arguments.

- However, proofs in formal logic can be rather long and complex to perform. Often a more informal proof is sufficient.

- In many situations we are not interested in the validity of an argument but if a conjecture is correct in a given interpretation.

- We have to be able to proof not only if something is correct but also if it is wrong. To be able to do this and to facilitate the proof of a large range of problems we need a variety of different proof techniques.

2 Informal Proofs

- In general we want to prove a conjecture $P \rightarrow Q$ in a specific context.

- Once a conjecture is proven it becomes a theorem.

- Conjectures are often created based on inductive reasoning. Inductive reasoning describes the process of drawing conclusions based on a number of experiences.

- To prove a conjecture we have to apply deductive reasoning where the truth or falsity is verified by evaluating or transforming the conjecture.

- Proofs are often presented in a narrative form, taking advantage of the properties of the specific context to allow shorter proofs.
  
  – Note: Informal proofs have to contain enough detail to be reproduceable and to be translated into a formal proof by introducing the rules specific to the context.

  – In informal proofs conjectures like $(\forall x)(P(x) \rightarrow Q(x))$ are mostly formulated as $P(x) \rightarrow Q(x)$ where the quantifier is maintained only implicitly. In formal logic this would be achieved by applying universal instantiation and then later universal generalization.
3 Proof Techniques

- Disproof by counterexample
  - To prove that a conjecture is false it is sufficient to find a single example which contradicts the conjecture.
  - Example: For every positive integer you can find a smaller positive integer.
    * There is no smaller positive integer than 1.
  - Example: For every positive integer \( n \), \( n! \leq n^2 \).
    * \( 4! > 4^2 \)
  - Note: There is no general way to find a specific counterexample. Moreover, there is no mechanical way to determine if a conjecture is true or false.

- Exhaustive proof
  - In an exhaustive proof the conjecture is shown true for every possible case.
  - This proof technique can only be applied if the set of cases is finite.
  - One example of an exhaustive proof technique are truth tables in propositional logic.
  - Example: Every even integer between 4 and 6 is the product of exactly 2 prime numbers.
    * \( 4 = 2 \times 2 \)
    * \( 6 = 2 \times 3 \)

- Direct proof
  - In a direct proof we assume the hypothesis \( P \) of a conjecture \( P \rightarrow Q \) and try to deduce the conclusion \( Q \).
  - Proof sequences in propositional or predicate logic are examples of direct proof techniques.
  - Example: The product of an even and an odd integer is even.
    * If \( N \) is an even integer then there is an integer \( k \) with \( n = 2 \times k \). The product of \( n \) with an odd integer \( m \) is thus \( n \times m = 2 \times k \times m \) and is therefore an even integer.

- Proof by contraposition
  - In this proof techniques the contrapositive \( \overline{Q} \rightarrow \overline{P} \) of the conjecture \( P \rightarrow Q \) is proven. Since the contrapositive is tautologically equivalent to the conjecture, such its proof is sufficient.
  - Example: If there are more than \( n \) assignment sheets given out to \( n \) students, then some student gets more than one assignment sheet.
∗ Contrapositive: If no student gets more than one assignment sheet, then no more than \( n \) assignment sheets are given out to \( n \) students.

Suppose each student gets \( \leq 1 \) assignment sheet. Then \( n \) students get \( m \leq n \times 1 \) assignment sheets.

– Note: The contrapositive is different from the converse of a conjecture. The converse \( Q \rightarrow P \) is not tautologically equivalent to the conjecture \( P \rightarrow Q \) and proving the converse does not prove the conjecture.

• Proof by contradiction

– Proof by contradiction uses again a transformation of the original conjecture \( P \rightarrow Q \). Here the argument \( P \land \neg Q \rightarrow F \) is proven. In other words the hypothesis and the negation of the conclusion are assumed and it has to be shown that that leads to a contradiction.

– Example: The sum of two even integers is not odd.

∗ Let us assume that the sum of two even integers is odd. This sum \( m+n = 2k+1 \).

However, \( m \) is an even integer and thus \( n = 2l \). Therefore \( m = 2k + 1 - 2l = 2(k - l) + 1 \). On the other hand, \( m \) is even and thus \( m = 2j \). From this we can deduce that \( 2j = 2(k - l) + 1 \) or \( 2(j + l - k) = 1 \), which is a contradiction. This proves the original conjecture.

• Proof by cases

– Proof by cases combines proof by exhaustion with some of the other proof techniques to simplify the overall proof. Here the domain is divided into an exhaustive set of cases which are proven independently. Each of these individual proofs can use a different proof technique.

– Example: The sum of two integers is odd if and only if one of the integers is odd and the other is even.

∗ Cases:

1. both integers are even: \( m+n = 2k + 2l = 2(k+l) \). Therefore the sum is even.
2. both integers are odd: \( m+n = 2k+1+2l+1 = 2(k+l)+2 = 2(k+l+1) \). Therefore the sum is even.
3. One integer is even and the other is odd: \( m+n = 2k+2l+1 = 2(k+l)+1 \). Therefore the sum is odd.

• Note: There is no general rule which indicates which proof technique should be used.

• Hint: Very often proof by contradiction is useful of the conclusion is a negation. Also proof by contraposition is often useful if the negation of the conclusion contains more, and more expressive conjunct than the original hypothesis. Before starting a proof, always look at the conjecture and think about which proof technique to use.

• Examples:
– If a number is one of 9, 6, 21, 39, then it is the product of exactly two prime numbers.
– The sum of two even numbers is divisible by 2.
– $\sqrt{2}$ is not a rational number
– In an elimination tournament with a total of 32 players the total number of games is 31.
– If the square of an integer is odd, then the integer is odd.
– The sum of two consecutive integers is odd