1

(a) Using the deduction method, we can rewrite the given argument as:

$$[(\forall x)[P(x) \vee Q(x)] \wedge [(\exists x)P(x)]'] \rightarrow (\forall x)Q(x)$$

Consider the following proof sequence:

(i) $[(\exists x)P(x)]'$ hypothesis.

(ii) $P(a)'$ (i), e.i.
(iii) $(\forall x)[P(x) \vee Q(x)]$ hypothesis.
(iv) $P(a) \vee Q(a)$ (iii), u.i.
(v) $P(a)' \vee Q(a)$ (iv), implication.
(vi) $Q(a)$ (ii), (v), Modus Ponens.
(vii) $(\forall x)Q(x)$ u.g.

The last step is justified, since $Q(a)$ was not deduced from a hypothesis in which a is a free variable nor has $Q(a)$ been deduced by existential instantiation from a formula in which a is a free variable.

(b) Let $M(x) \equiv$ "x is a member of the board", $G(x) \equiv$ "x comes from Government", $I(x) \equiv$ "x comes from Industry", $F(x) \equiv$ "x is in favor of the motion" and $L(x) \equiv$ "x has a law degree".
Accordingly, the given argument can be symbolized as follows:

$$[(\forall x)(M(x) \rightarrow (G(x) \vee I(x))) \wedge (\forall x)((G(x) \wedge L(x)) \rightarrow F(x)) \wedge [I(John)]' \wedge L(John)] \\ \rightarrow (M(John) \rightarrow F(John))$$

We use the Deduction Method to rewrite the above argument as:

$$[(\forall x)(M(x) \rightarrow (G(x) \vee I(x))) \wedge (\forall x)((G(x) \wedge L(x)) \rightarrow F(x)) \wedge [I(John)]' \wedge L(John) \wedge M(John)] \\ \rightarrow F(John)$$

Consider the following proof sequence:

(i) $\forall x(M(x) \rightarrow (G(x) \vee I(x)))$ hypothesis.
(ii) $M(John) \rightarrow (G(John) \vee I(John))$ Universal Instantiation.
(iii) $M(John)$ hypothesis.
(iv) $G(John) \vee I(John)$ (ii), (iii), Modus Ponens.
(v) $[I(John)]' \rightarrow G(John)$ (iv), implication equivalence.
(vi) $[I(John)]'$ hypothesis.
(vii) $G(John)$ (v), (vi), Modus Ponens.
(viii) $L(John)$ hypothesis.
(ix) $G(John) \wedge L(John)$ (vii), (viii), Conjunction.
(x) $(\forall x)((G(x) \wedge L(x)) \rightarrow F(x))$ hypothesis.
(xi) $(G(John) \wedge L(John)) \rightarrow F(John)$ (x), Universal Instantiation.
(xii) $F(John)$ (ix), (xi), Modus Ponens.

2

Solution: Let x and $(x+1)$ denote two consecutive integers. Observe that,

$$\begin{aligned} (x+1)^3 - x^3 &= [x^3 + 3 \cdot x^2 + 3 \cdot x + 1] - x^3 \\ &= 3 \cdot x \cdot (x+1) + 1 \end{aligned}$$

Regardless of whether x is even or odd, $x \cdot (x+1)$ is even and so is $3 \cdot x \cdot (x+1)$. It follows that $3 \cdot x \cdot (x+1) + 1$ is odd. □

3

Solution: We need to show that if $x+1$ is negative (or zero), then x is negative. Observe that decrementing a negative number always results in a negative number. This if $(x+1) < 0$, them $(x+1) - 1 = x < 0 - 1 = -1$. Likewise, if $x + 1 = 0$, then $x = -1 < 0$. The claim follows. □

4

Solution: Let x denote an integer and x^2 denote its square. If x is odd, then so is x^2 and hence $x + x^2$ is even. If x is even, then so is x^2 and hence $x + x^2$ is even.

In either case, $(x + x^2)$ is even and the claim follows. □

5

Solution: BASIS: $n = 2$.

$$\begin{aligned} LHS &= \sum_{i=1}^{2} \frac{1}{i^2} \\ &= \frac{1}{1} + \frac{1}{4} \\ &= \frac{5}{4} \\ &< \frac{3}{2} \\ &= 2 - \frac{1}{2} \\ &= RHS \end{aligned}$$

Thus the basis is proven.

INDUCTIVE STEP: Assume that $\sum_{i=1}^{k} \frac{1}{i^2} < 2 - \frac{1}{k}$, for some $k \geq 2$. Observe that,

$$\begin{aligned} \sum_{i=1}^{k+1} \frac{1}{i^2} &= \sum_{i=1}^{k} \frac{1}{i^2} + \frac{1}{(k+1)^2} \\ &< (2 - \frac{1}{k}) + \frac{1}{(k+1)^2}, \quad \text{by inductive hypothesis} \\ &= 2 - (\frac{1}{k} - \frac{1}{(k+1)^2}) \\ &= 2 - (\frac{(k+1)^2 - k}{k(k+1)^2}) \\ &= 2 - (\frac{k^2 + 2k + 1 - k}{k(k+1)^2}) \\ &= 2 - (\frac{k^2 + k + 1}{k(k+1)^2}) \\ &< 2 - (\frac{k^2 + k}{k(k+1)^2}), \quad \text{subtracting a smaller number} \\ &= 2 - \frac{k(k+1)}{k(k+1)^2} \\ &= 2 - \frac{1}{k+1} \end{aligned}$$

We have thus shown that, if the conjecture is true for any $k \geq 2$, then it must also be true for $(k + 1)$. Using the first principle of mathematical induction, we conclude that the presented conjecture is true for all $n \geq 2$. □