1. Show using induction that

$$\sum_{i=1}^{n}[F(i)]^2 = F(n)\cdot F(n+1).$$

Solution: BASIS: At $n = 1$, we have,

$$\begin{aligned} LHS &= [F(1)]^2 \\ &= 1^2 \\ &= 1 \end{aligned}$$

Likewise,

$$\begin{aligned} RHS &= F(1)\cdot F(2) \\ &= 1\cdot 1 \\ &= 1 \end{aligned}$$

Since $LHS = RHS$, the basis is proven.

INDUCTIVE STEP: Assume that the hypothesis holds for some $k \geq 1$, i.e., assume that

$$\sum_{i=1}^{k}[F(i)]^2 = F(k)\cdot F(k+1).$$

Observe that,

$$\begin{aligned} \sum_{i=1}^{k+1}[F(i)]^2 &= \sum_{i=1}^{k}[F(i)]^2 + [F(k+1)]^2 \\ &= F(k)\cdot F(k+1) + F(k+1)^2, \ \text{ by the inductive hypothesis} \\ &= F(k+1)\cdot[F(k)+F(k+1)] \\ &= F(k+1)\cdot F(k+2), \ \text{ definition of Fibonacci sequence} \end{aligned}$$

By applying the first principle of mathematical induction, we conclude that the conjecture is true for all integers n.

□

2. How many distinct *binary* operations can be defined on a set of n elements?

   Solution: Focus on a specific binary operation (say $\oplus$) defined on the n elements. In order to define this operation, we need to construct a table with $n \times n$ entries such that the $(i,j)^{th}$ entry corresponds to $a_i \oplus a_j$, where a_i and a_j are the i^{th} and j^{th} element of the set respectively. Now note that for a binary operation that is distinct from $\oplus$, at least one of the $n \times n = n^2$ entries has to be different. Thus, the total number of binary operations corresponds to the number of distinct ways of filling up the operator table. Since there are n choices for each of the entries, we apply the multiplication principle to conclude that the total number of distinct binary operators is n^{n^2}.

   □

3. Show that the set of all *infinite* length strings over the alphabet $\{a, b\}$ is not countable.

   Solution: The proof closely mirrors the proof discussed in class to prove that the set of real numbers is not countable.

   Assume that the given set (say S) is countable; it follows that the set is denumerable and therefore there exists some enumeration of S. Let

$$\begin{array}{c} s_{11}s_{12}s_{13}\cdots \\ s_{21}s_{22}s_{23}\cdots \\ \vdots \\ s_{p1}s_{p2}s_{p3}\cdots \\ \vdots \end{array}$$

   denote one such enumeration. Construct the following string $s = s_1 s_2 \ldots$ as follows: $s_i = a$, if $s_{ii} = b$, else $s_i = b$. Clearly, s is an infinite length string over $\{a, b\}$ and hence belongs to S. But it cannot be the first string since it differs from the first string in the first position; likewise, it differs from the second string in the second position and in general it differs from every string in the enumeration in at least one position. Thus s cannot belong to S; this contradiction arose because we assumed that S is countable. It follows that S is not countable.

   □

4. Let A and B denote two arbitrary sets and let $\mathcal{P}(A)$ and $\mathcal{P}(B)$ denote their power sets respectively. Argue that,

$$\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$$

Solution: Let $C = A \cap B$. Observe that $\mathcal{P}(C)$ is a set of sets, with each set being a subset of C. Let S denote an arbitrary element of $\mathcal{P}(C)$, i.e., $S \in \mathcal{P}(C)$. By the definition of power sets, $S \subseteq C$ and hence $S \subseteq A \cap B$. It follows that $S \subseteq A$ **and** $S \subseteq B$. By the definition of power sets, $S \in \mathcal{P}(A)$ and $S \in \mathcal{P}(B)$. Hence, $S \in \mathcal{P}(A) \cap \mathcal{P}(B)$.

Let $S \in \mathcal{P}(A) \cap \mathcal{P}(B)$. Therefore, $S \in \mathcal{P}(A)$ **and** $S \in \mathcal{P}(B)$. Since $S \in \mathcal{P}(A)$, $S \subseteq A$; likewise, $S \subseteq B$. Therefore, $S \subseteq A \cap B$. Therefore, $S \in \mathcal{P}(A \cap B)$. □

5. Let A, B and C denote three arbitrary sets. Show that

(a)

$$(A \cup B) - C = (A - C) \cup (B - C)$$

(b)

$$[(A' \cup B') \cap A']' = A$$

Solution:

(a) Observe that,

$$\begin{aligned}
x \in (A \cup B) - C & \\
& \rightarrow \quad x \in (A \cup B) \textbf{ and } x \notin C \\
& \rightarrow \quad (x \in A \textbf{ or } x \in B) \textbf{ and } x \notin C \\
& \rightarrow \quad (x \in A \textbf{ and } x \notin C) \textbf{ or } (x \in B \textbf{ and } x \notin C) \\
& \rightarrow \quad (x \in A - C) \textbf{ or } (x \in B - C) \\
& \rightarrow \quad x \in (A - C) \cup (B - C)
\end{aligned}$$

We have thus shown that $[(A \cup B) - C] \subseteq [(A - C) \cup (B - C)]$. The above argument can be reversed to show that $[(A - C) \cup (B - C)] \subseteq [(A \cup B) - C]$.

(b) First observe that

$$\begin{aligned}
(A' \cup B') \cap A' &= (A' \cap A') \cup (A' \cap B'), \text{ Distributivity} \\
&= A' \cup (A' \cap B'), \text{ Identity} \\
&= A', \text{ since } A' \cap B \subseteq A
\end{aligned}$$

Therefore, $[(A' \cup B') \cap A']' = (A')' = A$, as per the rules of complementation.

□