Discrete Structures

CSE 2315 (Spring 2014)

Lecture 11 Recursive and Sequence

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Midterm Exam

- Prove propositional logic arguments valid
- Prove predicate logic arguments valid
- Direct proof
- Contradiction proof
- Inductive proof

Recursive Definitions

- A recursive definition (or inductive definition) is one, in which the object being defined is part of the definition.
- Parts of a recursive definition
  - A basis, where some simple cases of the object being defined are explicitly provided,
  - An inductive or recursive step, where new cases of the item being defined are given in terms of previous cases.
- Strong connection between induction and recursion.
- Recursive Objects
  - Sequences.
  - Sets.
  - Operations.

Sequences

- A sequence is a list of objects that is enumerated in some order.

- Example: write down the first 5 elements of the following recursively defined sequence:
  - $S(1) = 2$
  - $S(n) = 2\,S(n - 1)$, $n >= 2$
  - The second part of the definition is called a recurrence relation.

- Example: write down the first 5 elements of the following recursively defined sequence:
  - $T(1) = 1$
  - $T(n) = T(n - 1) + 3$, $n >= 2$

Sequences

- Fibonacci Sequence
  - $F(1) = 1$
  - $F(2) = 1$
  - $F(n) = F(n - 1) + F(n - 2), n >= 3$

- Example: enumerate the first 5 elements of the Fibonacci sequence. Show that $F(n + 4) = 3 F(n + 2) - F(n)$, for all $n >= 1$
  - Induction proof
  - Direct proof

Recursively Defined Sets

- Define the set of ancestors of John.
  - John's parents are his ancestors.
  - If x is an ancestor of John and y is the parent of x, then y is an ancestor of John.

- Define the set of all possible word combinations using small-case letters from the English alphabet.
  - The empty string λ is a word.
  - {**a, b, c, . . . , z**} are words.
  - If x and y are words, then so is xy

Recursively Defined Operations

- Certain binary operations can be defined recursively in terms of “less powerful” operations

- Define exponentiation in terms of multiplication.
  - $a^0 = 1$
  - $a^m = a\ (a^{m-1}),\ m>=1$

- Define multiplication in terms of addition.
  - $x * 0 = 0$
  - $x\ y = x + x\ (y - 1),\ y >= 1$

Solving recurrences

- Two methods
  - Expand-Guess-Verify (EGV).
  - Formula.

- Example

  Consider the recurrence:

  $S(1) = 2$

  $S(n) = 2 \cdot S(n - 1)$, $n >= 2$.

- Example

  Solve the recurrence:

  $T(1) = 1$

  $T(n) = T(n - 1) + 3$, $n >= 2$.

Formula Approach

- A general linear recurrence has the form:

  $S(n) = f_1(n) \cdot S(n-1) + f_2(n) \cdot S(n-2) + \ldots f_k(n) \cdot S(n-k) + g(n)$

- The above formula is called linear, because the S() terms occur only in the first power. It is called first-order, if S(n) depends only on $S(n-1)$.

  For example, $S(n) = c \cdot S(n-1) + g(n)$. The recurrence is called homogeneous, if $g(n) = 0$, for all n.

- Formula for Linear first-order recurrence

$$\begin{aligned} S(1) &= k_0 \\ S(n) &= c \cdot S(n-1) + g(n) \end{aligned}$$

$$\Rightarrow S(n) = c^{n-1} \cdot k_0 + \sum_{i=2}^{n} c^{n-i} \cdot g(i).$$

Second Order homogeneous Linear Recurrence with constant coefficients

- Form: $S(n) = c_1 \cdot S(n-1) + c_2 \cdot S(n-2)$, subject to some initial conditions.
- Solve the characteristic equation: $t^2 - c_1 \cdot t - c_2 = 0$. Let r_1 and r_2 denote the roots.
- If $r_1 \mathrel{!}= r_2$, solve
  $p + q = S(1)$
  $p \cdot r_1 + q \cdot r_2 = S(2)$
  Then, $S(n) = p \cdot r_1^{n-1} + q \cdot r_2^{n-1}$
- If $r_1 = r_2 = r$, solve
  $p = S(1)$
  $(p + q) \cdot r = S(2)$
  Then, $S(n) = p \cdot r^{n-1} + q \cdot (n-1) \cdot r^{n-1}$

Examples of second order recurrences

- Solve the recurrence relation

  $T(1) = 5$

  $T(2) = 13$

  $T(n) = 6\cdot T(n-1) - 5\cdot T(n-2)$, $n >= 3$

- Solve the recurrence relation:

  $S(1) = 1$

  $S(2) = 12$

  $S(n) = 8\cdot S(n-1) - 16\cdot S(n-2)$, $n >= 3$