Discrete Structures

CSE 2315 (Spring 2014)

Lecture 13 Counting

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Multiplication Principle

- If there are n_1 possible outcomes for a first event and n_2 possible outcomes for a second event, then there are $n_1 * n_2$ possible outcomes for the sequence of the two events.
- Example

  How many 4 digit numbers are there? (Include those which begin with 0)

  Solution:

  There are 10 ways to select the first digit, 10 ways to select the second digit, 10 ways to select the third digit and 10 ways to select the fourth digit. As per the multiplication principle, there are 10 x 10 x 10 x 10 = 10000 ways to construct a 4 digit number.

Examples of the Multiplication Principle

- How many 4 digits are there, if no digit can be repeated?

  Solution: 10 x 9 x 8 x 7 = 5040.

- Let A and B denote two sets. How many elements does A B have?

  Solution: Using the multiplication principle, |A|x |B|.

Addition Principle

- If A and B are disjoint events with n_1 and n_2 possible outcomes respectively, then the total number of possible outcomes for the event "A **or** B" is $n_1 + n_2$.

- Example
  - A customer wishes to purchase a vehicle from a dealer. The dealer has 10 trucks and 5 cars. How many choices does the customer have?

    Solution: $10 + 5 = 15$.
  - Let A and B be two disjoint sets. What is $|A \cup B|$?

    Solution: By the addition principle, $|A| + |B|$.
  - Let A and B denote two finite sets; show that
  - $|A - B| = |A| - |A \cap B|$

    Solution: The key observation is that $(A - B)$ and $A \cap B$ are disjoint. Further, the union of $(A - B)$ and $A \cap B$ is A! Therefore, by the addition principle, $|A| = |A - B| + |A \cap B|$.

Combining addition and multiplication principles

- How many 4 digit numbers begin with a 4 or a 5?

  Solution: Using the multiplication principle, the number of 4 digit numbers which begin with 4 is 1 x 10 x 10 x 10 = 1000. Likewise, the number of 4 digit numbers which begin with 5 is 1000. Thus the number of 4 digit numbers which begin with a 4 or a 5 is 1000 + 1000 = 2000, using the addition principle.
- How many 3 digit numbers between 100 and 999 (inclusive) are even?

  Solution: Every even number ends in 0, 2, 4, 6 or 8. Use multiplication principle to compute the number of even numbers that end in 0, that end in 2 and so on (9 x 10 x 1). Use the addition principle to get the total number of even numbers (450).
- How many 4 digits are there in which at least one digit is repeated?

  Solution: Find the total number of 4 digit numbers and subtract the 4 digit numbers with no repetitions!

Decision Trees

- The multiplication principle cannot be used if the number of choices at a given stage depends upon the exact choice made in the previous stage.

- Examples:
  - In the 4 digit problems, it did not matter what digit was chosen in the current stage.
  - In how many ways can you toss a coin 5 times, so that the head side does not show up in consecutive tosses?

Pigeonhole Principle

- If more than k items are placed in k bins, then at least one bin contains more than one item.
- How many times should a die be tossed before you can be certain that the same value shows up twice?

  Solution: 7.
- Show that if 51 positive integers between 11 and 100 are chosen, then one of them must divide the other.

  Solution: Every number can be expressed as a product of prime numbers. Let $n_1, n_2, \ldots, n_{51}$ denote the chosen numbers. Therefore, each $n_i = 2^{ki} \cdot b_i$, where b_i is some odd number, such that $1 <= b_i <= 99$.

  But there are exactly 50 odd numbers between 1 and 99.

  Therefore, $b_i = b_j$, for some pair (n_i, n_j) (pigeonhole principle).

  In other words, we must have $n_i = 2^{ki} \cdot b_i$ and $n_j = 2^{kj} \cdot b_j$.

  Depending on whether $k_i >= k_j$ or vice versa, one of n_i and n_j must divide the other.

Inclusion and Exclusion

- Give a formula for $|A \cup B|$ in terms of $|A|$, $|B|$ and $|A \cap B|$?
- Give a formula for $|A \cup B \cup C|$ in terms of $|A|$, $|B|$, $|C|$ and related sets?
- Solution

$$|A \cup B| = |A| + |B| - |A \cap B|.$$
$$A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|.$$

- The Inclusion-Exclusion Principle

$$|A_1 \cup A_2 \cup \ldots A_n| = \sum_{1 \le i \le n} |A_i| - \sum_{1 \le i < j \le n} |A_i \cap A_j| + \sum_{1 \le i < j < k \le n} |A_i \cap A_j \cap A_k| + \ldots (-1)^{n+1} |A_1 \cap A_2 \ldots A_n|$$

- Can be proved using induction. Beyond scope of course.

Applying the Inclusion-Exclusion Principle

- Consider a group of students ordering pizza. 13 will eat sausage topping, 10 will eat pepperoni, 12 will eat pineapple, 4 will eat both sausage and pepperoni, 5 will eat both pepperoni and pineapple, 7 will eat both sausage and pineapple and 3 will eat all three toppings. How many students are there in the group?
- Solution

Let $A \equiv$ students who eat sausage, $B \equiv$ students who eat pepperoni and $C \equiv$ students who eat pineapple. Accordingly,

$|A| = 13,\ |B| = 10,\ |C| = 12,\ |A \cap B| = 4,\ |B \cap C| = 5,\ |A \cap C| = 7, |A \cap B \cap C| = 3.$

Using the Inclusion-Exclusion principle,

$$|A \cup B \cup C| = 13 + 10 + 12 - (4 + 5 + 7) + 3 = 22$$

One More Example

- 35 voters were queried about their opinions regarding two referendums. 14 supported referendum 1 and 26 supported referendum 2. How many voters supported both, assuming that every voter supported either referendum 1 or referendum 2 or both?
- Solution

Let $A \equiv$ voters who supported referendum 1 and $B \equiv$ voters who supported referendum 2. Then, we have, $|A \cup B| = 35$, $|A| = 14$ and $|B| = 26$. Using the Inclusion-Exclusion principle,

$$\begin{aligned} |A \cup B| &= |A| + |B| - |A \cap B| \\ \rightarrow |A \cap B| &= |A| + B| - |A \cup B| \\ &= 14 + 26 - 35 \\ &= 5 \end{aligned}$$

Permutations

- A permutation is an ordered arrangement of objects. The number of distinct permutations of r distinct objects chosen from n distinct objects is denoted by P(n, r).
- n! = 1, n = 0; n! = n (n - 1)!, otherwise
- Computing P(n, r)

  Using the multiplication principle

$$
\begin{aligned}
P(n,r) &= n \cdot (n-1) \cdot \ldots (n-r+1) \\
&= n \cdot (n-1) \cdot \ldots (n-r+1) \cdot \frac{(n-r) \cdot (n-r-1) \cdot \ldots 1}{(n-r) \cdot (n-r-1) \cdot \ldots 1} \\
&= \frac{n!}{(n-r)!},\ 0 \leq r \leq n
\end{aligned}
$$

Permutations

- Compute P(7, 3), P(n, 0), P(n, 1), and P(n, n).

  Solution: 210, 1, n, and n!.

- How many 3 letter words can be formed using the letters in the word "compiler"?

  Solution: P(8, 3).

- In how many ways can a president and vice-president chosen from a group of 20 people?

  Solution: P(20, 2).

Example

A library has 4 books on programming, 7 on algorithms and 3 on complexity. In how many ways can the books be ordered on a shelf?

Provided that the books of a subject are required to be together?

Solution: If there is no restriction, the number of arrangements is $P(14, 14) = 14!$. Now consider the case in which the books of a given subject are required to be together. First arrange the three subjects. This can be done in $P(3, 3) = 3!$ ways. Corresponding to each such arrangement, the programming books can be permuted in $P(4, 4) = 4!$ ways, the algorithms books can be permuted in $P(7, 7) = 7!$ ways and the complexity books can be permuted in $P(3, 3) = 3!$ ways. Using the multiplication principle, the total number of arrangements is $3! \times 4! \times 7! \times 3!$.

Combinations

- A combination is an (unordered) selection of objects. The number of distinct combinations of r distinct objects chosen from n distinct objects is denoted by C(n, r).

- Computing C(n, r)

  Focus on a given combination of r objects chosen from n objects. The objects in this combination can be permuted in r ! different ways to get r ! distinct permutations. It follows that C(n, r) r ! = P(n, r), i.e., C(n, r) = P(n, r) / r ! = n! / r !(n - r)! ; 0 <= r <= n.

- Compute C(7, 3), C(n, 0), C(n, 1) and C(n, n).

  Solution: 35, 1, n, 1.

Combinations

- A committee of 8 students is to be selected from 19 freshmen and 34 sophomores. In how many ways, can this committee be formed, if
  - it must contain 3 freshmen and 5 sophomores.

    Solution: C(19, 3)C(34, 5).
  - it must contain exactly one freshman.

    Solution: C(19, 1)C(34, 7).
  - it can contain at most one freshman.

    Solution: C(34, 8) + C(19, 1)C(34, 7).
  - it contains at least one freshman.

    Solution: C(53, 8) – C(34, 8).

Handling Duplicates

- Permutations

  If there are n objects of which a set of n_1 are indistinguishable from each other, a second set of n_2 are indistinguishable from each other and . . . a k^{th} set of n_k objects are indistinguishable from each other, then the number of distinct permutations of the n objects is:

  n!/(n1! n2! … nk !)

- In how many distinct ways can the characters in the word MISSISSIPPI be permuted?

  Solution: 11!/4!4!2! .

Handling Repetitions

- Permutations

  The number of arrangements of r objects from n objects with repetitions permitted is simply n^r .

- Combinations

  The number of combinations of r objects selected from n objects with repetitions allowed is $C(r + n - 1, r)$.

- Example

  In how many ways can six children choose one lollipop from a selection of red, green and yellow lollipops, assuming that we do not care which child gets which?

  Solution: $C(6 + 3 - 1, 6) = C(8, 6)$.