Discrete Structures

CSE 2315 (Spring 2014)

Lecture 15 Relations

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Fundamental Notions

- Given a set S, a **binary** relation on a set S is any subset of S x S, , i.e., any set of ordered pairs of elements of S. We typically use x ρ y to mean (x, y) $\in$ ρ.

- Example:

  Let S = {1, 2}. S x S = {(1, 1), (1, 2), (2, 1), (2, 2)}. Let ρ be a relation on S x S, defined as follows: x ρ y $\Leftrightarrow$ x + y is odd. Then, ρ = {(1, 2), (2, 1)}.

- Given n sets S1, S2, … , Sn, an n-ary relation on S1 x S2 … Sn is any subset of S1 x S2 … Sn.

Examples

- Membership

  Let ρ be a relation on N x N defined as: $x \rho y \leftrightarrow x = y + 1$. Enumerate the elements of ρ.

  Solution: (1, 0), (2, 1), (3, 2), ...

- A binary relation on A x B is a **pairing** of elements in A, with the elements in B.

Classification

- Let ρ be a binary relation defined on S X T. Observe that each element of ρ has the form (s1, s2), where s1∈S and s2∈T. ρ is said to be:
  - **one-one**, if each first component and each second component appear exactly once, e.g., ρ = {(1, 2), (2, 1)}.
  - **one-many**, if some first component appears more than once, e.g., ρ = {(1, 1), (1, 2)}.
  - **many-one**, if some second component, appears more than once, e.g., ρ = {(1, 1), (2, 1)}.
  - **many-many**, if some first component appears more than once and some second component appears more than once, e.g., ρ = {(1, 1), (2, 1), (1, 3)}.

Set Properties

- Relations **are** sets; therefore, all set identities (commutativity, associativity, distributivity, etc.) also apply to relations. In particular, $\rho \cup \rho' = S^2$ and $\rho \cap \rho' = \emptyset$

- Additional Properties

A relation ρ on $S \times S$ is said to be:

(i) **Reflexive**, if $(\forall x)(x \in S \rightarrow (x, x) \in \rho)$.

(ii) **Symmetric**, if $(\forall x)(\forall y)(x \in S \wedge y \in S \wedge (x, y) \in \rho \rightarrow (y, x) \in \rho)$.

(iii) **Transitive**, if $(\forall x)(\forall y)(\forall z)(x \in S \wedge y \in S \wedge z \in S \wedge (x, y) \in \rho \wedge (y, z) \in \rho \rightarrow (x, z) \in \rho)$.

(iv) **Antisymmetric**, if $(\forall x)(\forall y)(x \in S \wedge y \in S \wedge (x, y) \in \rho \wedge (y, x) \in \rho \rightarrow x = y)$.

Examples

- Relations on N x N
  - = is reflexive, symmetric and transitive.
  - < is transitive but not reflexive or symmetric.
  - <= is antisymmetric.

- Relations on the power set P(S) of a set S

  The relation $\subseteq$ is antisymmetric

Closure of a relation

- A binary relation ρ^* on a set S, is the closure of a relation ρ on S with respect to a property P, if
  - ρ^* has property P,
  - $\rho \subseteq \rho^*$,
  - ρ^* is the subset of any other relation on S that includes ρ and has property P.

- Let S = {1, 2, 3} and ρ = {(1, 1), (1, 2), (1, 3), (3, 1), (2, 3)}.
  - Is ρ reflexive? The reflexive closure is: $\rho \cup$ {(2, 2), (3, 3)}.
  - Is symmetric? The symmetric closure is: $\rho \cup$ {(2, 1), (3, 2)}.
  - Is transitive? The transitive closure is: $\rho \cup$ {(3, 2), (3, 3), (2, 1), (2, 2)}.

- Compute the reflexive and transitive closure of ρ.

Partial Orderings

- A binary relation on a set S that is reflexive, antisymmetric and transitive is called a **partial ordering** on S.

- Example

  (i) On $\mathcal{N}$, $x \,\rho\, y \leftrightarrow x \leq y$.

  (ii) On $\mathcal{P}(\mathcal{N})$, $A \,\rho\, B \leftrightarrow A \subseteq B$.

  (iii) On $\{0, 1\}$, $x \,\rho\, y \leftrightarrow x = y^2$.

- If ρ is a partial ordering on S, (S, ρ) is called a partially ordered set (or poset). (S, $\leq$) will be used to denote an arbitrary partially ordered set.

Partial Orderings

- Let (S, $\leq$) denote some poset. Let x and y be two elements in S, such that x $\leq$ y, but x != y (written as x < y). x is said to be a predecessor of y and y is said to be a successor of x. If there is no z $\in$ S, such that x < z < y, then x is said to be an immediate predecessor of y.

- If S is finite, the poset (S, $\leq$) can be represented by a **Hasse diagram**, in which elements are represented by vertices and the property "is-related-to" by a straight line.

Example

- Consider the relation $x \mid y$ on the set $S = \{1, 2, 3, 6, 12, 18\}$.
  - Enumerate the ordered pairs of the relation.

    Solution: {(1, 2), (1, 3), (1, 6), (1, 12), (1, 18), (2, 6), (2, 12), (2, 18), (3, 6), (3, 12), (3, 18), (6, 12), (6, 18), (1, 1), (2, 2), (3, 3), (6, 6), (12, 12), (18, 18)}.
  - Write all the predecessors of 18.

    Solution: {1, 2, 3, 6}.
  - Write the immediate predecessors of 6.

    Solution: {2, 3}.
  - Draw the Hasse diagram for this poset.

Additional Issues

- If every two elements of the ground set are related to each other, the partial ordering is called a total ordering or **chain**. e.g., $\leq$ on N.
- An element x $\in$ S is said to be minimal in the poset (S, $\leq$), if there is no element y such that y < x.
- An element x $\in$ S is said to be the least element in the poset (S, $\leq$), if for every element y $\in$ S, x $\leq$ y.
- If a poset (S, $\leq$) has a least element, then this element is unique and minimal. Every minimal element is not necessarily a least element.

Equivalence Relations

- A binary relation on a set S that is reflexive, symmetric and transitive is said to be an equivalence relation

- Example
  - On any set S, $x \rho y \leftrightarrow x = y$.
  - On N, $x \rho y \leftrightarrow x + y$ is even.

- A partition of a set S is a collection of nonempty disjoint sets whose union is S.

- We use [x] to denote the set $\{y \mid y \in S \wedge x \rho y\}$. [x] is said to be the equivalence class of x.

Partition theorem

- An equivalence relation ρ on a set S determines a partition of S and every partition of a set S determines an equivalence relation on S.

  Proof: Somewhat tedious but the main idea is that if there is an element common to two distinct equivalence classes, then these classes coincide.

- How does the equivalence relation $x \rho y \leftrightarrow x + y$ is even partition N?

  Solution: All odd numbers are in one partition and all even numbers in the other partition!

One more example

- For integers x and y and any positive integer n,
  x = y mod n, if x - y is an integral multiple of n

- Enumerate the equivalence classes of congruence modulo 4.
  Solution

$$[0] = \{\ldots, -8, -4, 0, 4, 8, \ldots\}$$
$$[1] = \{\ldots, -7, -3, 1, 5, \ldots\}$$
$$[2] = \{\ldots, -6, -2, 2, 6, \ldots\}$$
$$[3] = \{\ldots, -5, -1, 3, 7, \ldots\}$$