Discrete Structures

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Lecture 16 Functions

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Functions

- Some common functions

  $y = x^2$.

  $y = x^{1/2} - \sin x$.

- Definition

  Let S and T denote two sets. A function (mapping) from S to T (denoted by $f : S \rightarrow T$) is a subset of S X T, in which each member of S appears exactly once as the first component of an ordered pair. S is called the **domain** and T is called the **codomain** of the function. If $(s, t) \in f$, then we write $t = f(s)$; t is the image of s under f and s is the pre-image of t under f. For $A \subseteq S$, $f(A) = \{f(a) : a \in A\}$.

Examples

- Is $f : N \rightarrow N$ a function, where $f(x) = x^{1/2}$.
- Is $f : N \rightarrow R$ a function, where $f(x) = x^{1/2}$.
- Is $g : N \rightarrow N$ a function, where

  $g(x) = x$, if $x <= 5$

  $= 10 - x$, if $5 <= x <= 10$

  $= x$, if $x >= 11$

- Functions can be defined on more than one variable; for instance, $f : N \times N \rightarrow N$, $z = x^2 + y^2$.

Function Fundamentals

- Definition

  Two functions f and g are said to be equal, if they have the same domain, the same co-domain and the same association of values in the co-domain with values in the domain.

- Example

  Let S = {1, 2, 3} and let T = {1, 4, 9}. Let $f : S \rightarrow T$ be defined as follows: f = {(1, 1), (2, 4), (3, 9)}. The function $g : S \rightarrow T$ is defined as follows:

$$g(n) = \frac{\sum_{k=1}^{n}(4k-2)}{2}$$

  Is f = g?

Onto Functions

- Let $f : S \rightarrow T$ denote an arbitrary function. Let $R = \{t \mid t \in T$ and $t = f(s)$, for some $s \in S\}$, i.e., $R = f(S)$. R is called the range of f; clearly $R \subseteq T$. If $R = T$, then f is called an **onto** (or **surjective**) function.
- Example
  - Is $f : R^+ \rightarrow R$ defined as $f(x) = x^{1/2}$ surjective?
  - Is $f : R^+ \rightarrow R^+$ defined as $f(x) = x^{1/2}$ surjective?
  - Is $f : R \rightarrow R$ defined as $f(x) = x^2$ surjective?
  - Is $f : R \rightarrow R$ defined as $f(x) = x^3$ surjective?
- In order to show that a function $f : S \rightarrow T$ is surjective,
  - Pick an arbitrary element $t \in T$.
  - Show that there exists some $s \in S$, such that $f(s) = t$.

One-one Functions

- A function $f : S \rightarrow T$ is said to be **one-one** or **injective**, if distinct elements in S have distinct images in T.

- Example
  - Is $f : R \rightarrow R$ defined as $f(x) = x^2$ injective?
  - Is $f : R \rightarrow R$ defined as $f(x) = x^3$ injective?

- In order to show that a function $f : S \rightarrow T$ is injective,
  - Show that for arbitrarily chosen $a, b \in S$, $a \mathrel{!=} b \rightarrow f(a) \mathrel{!=} f(b)$.
  - Alternatively, show that for arbitrarily chosen $f(a), f(b) \in T$, $f(a) = f(b) \rightarrow a = b$.

Bijections

- Definition

  A function $f : S \rightarrow T$ is said to be **bjiective** if it is both injective and surjective.

- Example
  - Is $f : R \rightarrow R$ defined as $f(x) = x^2$ bijective?
  - Is $f : R \rightarrow R$ defined as $f(x) = x^3$ bijective?

Function Composition

- Let $f: S \rightarrow T$ and $g: T \rightarrow U$ denote two functions. Then, the composition function, $g \circ f$ is a function from S to U defined as $(g \circ f)(s) = g(f(s))$.

- Arbitrary functions cannot be composed; the domains and ranges have to be compatible.

- Let $f : R \rightarrow R$ be defined by $f(x) = x^2$ and $g: R \rightarrow R$ be defined by $g(x) = \lfloor x \rfloor$. Compute $(g \circ f)(2.3)$ and $(f \circ g)(2.3)$.

  Solution: $(g \circ f)(2.3) = g(f(2.3)) = g(5.29) = 5.$

  $(f \circ g)(2.3) = f(g(2.3)) = f(2) = 22 = 4.$

Function Composition

- The composition of two injective functions is injective.

  Proof.

Let $f : S \rightarrow T$ and $g : T \rightarrow U$ denote two injective functions.
Let $s_1, s_2 \in S$. We need to show that if $(g \circ f)(s_1) = (g \circ f)(s_2)$, then $s_1 = s_2$. As per the hypothesis,

$$\begin{aligned}
(g \circ f)(s_1) &= (g \circ f)(s_2) \\
\Rightarrow g(f(s_1)) &= g(f(s_2)) \\
\Rightarrow f(s_1) &= f(s_2), \text{ since } g \text{ is injective} \\
\Rightarrow s_1 &= s_2, \text{ since } f \text{ is injective} \\
\Rightarrow (g \circ f) \text{ is injective} &
\end{aligned}$$

Function composition

- The composition of two surjective function is surjective.

  Proof.

Let $f : S \to T$ and $g : T \to U$ denote two onto functions. Consider the composition $(g \circ f) : S \to U$. Pick an arbitrary element $u \in U$. Since g is surjective, there exists a $t \in T$, such that $g(t) = u$. Since f is surjective, there exists an $s \in S$, such that $f(s) = t$. In other words, $(g \circ f)(s) = u$. Since u was arbitrarily chosen, it follows that $(g \circ f)$ is surjective.

- The composition of two bijective functions is a bijective function.

Inverse Functions

- The function i_S: S → S which maps each element of S to itself, is called the identity function on S.

- Let f: S → T denote a bijection. Since f is onto, corresponding to every element t $\in$ T, there is some element s $\in$ S, such that f(s) = t. Since f is injective, there is only one s such that f(s) = t. But this could be construed as the existence of a function g: T → S, i.e., g(t) = s. Note that g is also a bijective function! Observe that (g o f)(s) = g(f(s)) = g(t) = s, i.e., (g o f) = i_S. Similarly, (f o g) = i_T.

Inverse Functions

- Let $f: S \rightarrow T$ denote a function; if there exists a function $g: T \rightarrow S$, such that $(g \circ f) = i_S$ and $(f \circ g) = i_T$, then g is called the inverse function of f and denoted by f^{-1}.

- $f: S \rightarrow T$ is a bijection if and only if f^{-1} exists.

- Example

  Find the inverse of $f : R \rightarrow R$ defined as $f(x) = 3x + 4$.

Equivalent Sets

- A set S is equivalent to a set T, if there is a bijection f: S → T. Two sets that are equivalent have the same cardinality.
- Do Z and N have the same cardinality?
- Cantor: For any set S, S and P(S) are not equivalent.

  Proof:

Assume that S and $\mathcal{P}(S)$ are equivalent and let f be a bijection between S and $\mathcal{P}(S)$. Pick $s \in S$; $f(s) \in \mathcal{P}(S)$ and hence $f(s)$ is a set containing some members of S. Let $X = \{x \in S \mid x \notin f(x)\}$. Since $X \subseteq S$, $X \in \mathcal{P}(S)$. Therefore, $X = f(y)$, for some $y \in S$, since f is a bijection!
Is $y \in X$? If $y \in X$, then $y \notin f(y) = X$! If $y \notin X$, then $y \notin f(y)$ and hence, $y \in X$, by the definition of X! In either case, there is a contradiction, which proves that S and $\mathcal{P}(S)$ are not equivalent.

Permutation Functions

- A bijection f from a set A to itself is called a permutation function. Note that f has A as both its domain and its range.
- A permutation function represents a reordering of the set. e.g., A = {1, 2, 3, 4}, f = (1 2 3 4; 2 3 1 4)
- A cycle in a permutation of the form (a_1, a_2, …, a_k) represents the fact that element a_1 is mapped to a_2, a_2 is mapped to a_3, . . . , a_{k-1} is mapped a_k and a_k is mapped to a_1. e.g., f above can be represented as (1 2 3) o (4).
- A permutation function in which no element is mapped to itself is called a derangement.

Counting Functions

Let S and T be sets with cardinality m and n respectively. Let $f : S \to T$ denote a function from S to T.

(i) How many functions are possible from S to T, assuming no restriction on f? $n \cdot n \cdot \ldots n = n^m$.

(ii) How many functions are possible from S to T, assuming that f is injective? $n \cdot (n-1) \cdot (n-1) \ldots [n-(m-1)] = \frac{n!}{(n-m)!} = P(n,m)$.

(iii) How many functions are possible from S to T, assuming that f is surjective? No nice formula or easy answer. We count the number of non-onto functions and subtract this quantity from the total number of functions!

Order of Magnitude of Functions

- The order theory enables us to compare functions, just as the theory of arithmetic enables us to compare numbers. In case of functions, we are interested in rate of growth, i.e., does function f grow at a faster rate than function g?
- Note
  - Constants do not matter in rate of growth.
  - The starting point of measurement does not matter.
  - We only care about functions from $R_{>=0} \rightarrow R_{>=0}$.
- Example:
  - Which function grows faster: $100x^2$ or $1/10^6\ x^3$?
  - Which function grows faster: $x^2 - 10$ or $x + 10$?

Order of Magnitude

- Let f and g be functions mapping non-negative reals to non-negative reals. Then $f = O(g)$, if there exist constants c and n_0 such that for all $n >= n_0$, $f(x) <= c\, g(x)$.
- Let f and g be functions mapping non-negative reals to non-negative reals. Then $f = \Omega(g)$, if there exist constants c and n_0 such that for all $n >= n_0$, $f(x) >= c\, g(x)$.
- Let f and g be functions mapping non-negative reals to non-negative reals. Then $f = \Theta(g)$, if $f = O(g)$ and $g = O(f)$.
- If $f = O(g)$, either $f = \Theta(g)$ or $f = O(g)$. If $f = \Omega(g)$, either $f = \Theta(g)$ or $g = O(f)$.

Examples

- Let $f(x) = 2x^2 - 2$ and $g(x) = 1/100\ x^2 - 100$. $f = \Theta(g)$.
- Let $f(x) = 2x^2 - 2$ and $g(x) = 1/100\ x - 100$. $f = \Omega(g)$.
- Let $f(x) = 2x^2 - 2$ and $g(x) = 1/100\ x - 100$. $g = O(f)$.

Test to determine order

Let f and g denote two functions mapping non-negative reals to non-negative reals. Let $l = \lim_{x\to\infty} \frac{f(x)}{g(x)}$. Then

(i) If l is a positive constant, then $f = \Theta(g)$.

(ii) If $l = 0$, then $f = o(g)$.

(iii) If $l = \infty$, then $g = o(f)$.

If $\lim_{x\to\infty} f(x) = \infty$ and if $\lim_{x\to\infty} g(x) = \infty$, then

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}$$

The above rule is called L'Hospital's rule.

- Show that $x = O(x^2)$.
- Show that $x = O(x \log x)$.
- Show that $\log x = O(x)$.