Discrete Structures

CSE 2315 (Spring 2014)

Lecture 18 Final Exam Practice

Heng Huang, Ph.D.

Department of Computer Science and Engineering

Final exam questions distribution

- 30% from lectures before midterm exam
  - Logic (10pts)
  - Proof (10pts)
  - Induction (10pts)
- 70% from lectures after midterm exam
  - Counting (10pts)
  - Permutations and combinations (10pts)
  - Binomial theorem (10pts)
  - Set (10pts)
  - Relation (20pts)
  - Function (10pts)

Final Exam: Thursday, May 8, 2pm-4:30pm

Logic

- Consider the following verbal argument: *If the program is efficient, it executes quickly. Either the program is efficient, or it has a bug. However, the program does not execute quickly. Hence it has a bug.*

  Is the argument valid?

Solution: We use the following symbols:

(a) E - the program is efficient;

(b) Q - the program executes quickle;

(c) B - the program has a bug.

Accordingly, the argument can be expressed as:

$$[(E \to Q) \wedge (E \vee B) \wedge Q'] \to B$$

Logic

Consider the following proof sequence:

(i) $(E \rightarrow Q)$ hypothesis.

(ii) Q' hypothesis.

(iii) E' (i), (ii), Modus Tollens.

(iv) $(E \vee B)$ hypothesis.

(v) $(E' \rightarrow B)$ (iv), implication rule.

(vi) B, (iii), (v), Modus Ponens.

In other words, the argument is valid. $\square$

Proof

- Prove that $6^{1/2}$ is irrational.

Counting

- A phone number is a 7-digit sequence that does not start with 0.
- How many number we have if we don't consider their orders?
- How many number we have if we consider their orders, and without 0 as the starting number.

Permutations and combinations

Show that for any n and r, with $0 \leq r \leq n$,

$$C(n, 2) = C(r, 2) + C(n - r, 2) + r \cdot (n - r).$$

Solution: We provide a combinatorial proof for the above identity.

Note that the LHS represents the number of distinct ways of selecting 2 objects from n distinct objects. We break up the set of n objects into two disjoint sets, with one set containing r objects and the other set containing $n - r$ objects. Observe that 2 objects can be selected in one of the following distinct ways:

(a) Both objects from the set of r objects; this can be done in $C(r, 2)$ ways.

(b) Both objects from the set of $n - r$ objects; this can be done in $C(n - r, 2)$ ways.

(c) One object from the set of r objects and the other object from the set of $n - r$ objects; this can be done in $r \cdot (n - r)$ ways.

Thus, the total number of ways to select 2 objects out of n is $C(r, 2) + C(n - r, 2) + r \cdot (n - r)$, which proves the identity.

You are of course welcome to attempt a proof using induction or first principles!

Binomial theorem

$$(a+b)^n = \sum_{i=0}^{n} C(n,i)\ a^{n-i} \cdot b^i, \quad \forall n \geq 0.$$

- The multiplication of n items of (a+b)

Set

- Among a bank's 214 customers, 189 have checking accounts, 73 have regular savings accounts, 114 have money market savings accounts and 69 have both checking and regular savings accounts. No customer is allowed to have both regular savings and money market savings accounts. How many customers have both checking and money market savings accounts?

  Let

  i. A denote the set of people with checking accounts,
  ii. B denote the set of people with regular savings accounts, and
  iii. C denote the set of people with money market savings accounts.

  We have that,

  i. $|A \cup B \cup C| = 214$,
  ii. $|A| = 189$,
  iii. $|B| = 73$,
  iv. $|C| = 114$,
  v. $|A \cap B| = 69$, and
  vi. $|B \cap C| = 0$

Relations

1. (24 points) For each of the following relations, state whether they fulfill each of the 4 main properties - reflexive, symmetric, antisymmetric, transitive. Briefly substantiate each of your answers.

   (a) The coprime relation on $\mathbb{Z}$. ($a, b \in \mathbb{Z}$ are coprime if and only if $gcd(a, b) = 1$.)

   (b) Divisibility on $\mathbb{Z}$.

   (c) The relation T on $\mathbb{R}$ such that $a\ T\ b$ if and only if $ab \in \mathbb{Q}$.

(a)

It's definitely not reflexive, as no integer is coprime with itself except -1 and 1. It is symmetric because $\gcd(a, b) = \gcd(b, a)$, so $\gcd(a, b) = 1$ iff $\gcd(b, a) = 1$. Not antisymmetric — *every* coprime pair, such as (5,7) and (7,5), will show this. Not transitive — $\gcd(5, 7) = 1$, $\gcd(7, 10) = 1$, but $\gcd(5, 10) \neq 1$.

(b)

Not reflexive, since 0|0 is undefined.

Not symmetric, for example $2 \mid 4$ but $4 \nmid 2$. It not antisymmetric on $\mathbb{Z}$, since $a \mid -a$ and $-a \mid a$, although it would be antisymmetric if restricted to $\mathbb{N}$. It is transitive — if $a \mid b$ then $b = ka$ for some $k \in \mathbb{Z}$, and if $b \mid c$ then $c = lb$ for some $l \in \mathbb{Z}$, thus $c = (lk)a$ and $(lk) \in \mathbb{Z}$ so $a \mid c$.

(c)

Not reflexive, for example $\sqrt[4]{2}\sqrt[4]{2} = \sqrt{2}$ which is definitely not in $\mathbb{Q}$. Definitely symmetric since multiplication is commutative, $ab = ba$ always. Not antisymmetric, since $\sqrt{2}\sqrt{8} = \sqrt{8}\sqrt{2} = 4$ but $\sqrt{2} \neq \sqrt{8}$. Also not transitive — consider $a = \pi$, $b = \frac{1}{\pi}$, and $c = \pi$. $ab, bc \in \mathbb{Q}$ but $ac = \pi^2 \notin \mathbb{Q}$.

Function

For $n \in \mathbb{N}$, $z \in (\mathbb{Z} \setminus \mathbb{N})$, $z = -n - 1$. This maps 0 to -1, 1 to -2, etc. Both sets are infinite sequences of integers, one starting at 0 and increasing, and the other starting at -1 and decreasing, so this function will cover all elements of both sets. It is one-one because $-n_1 - 1 = -n_2 - 1$ implies $n_1 = n_2$.

For $n \in \mathbb{N}$, $z \in \mathbb{Z}$, $z = -1^n \times \lceil \frac{n}{2} \rceil$. Here $\lceil x \rceil$, known as the *ceiling function*, denotes the smallest integer that is greater that or equal to x. Thus 0 is mapped to 0, 1 to -1, 2 to 1, etc. As n increases along the number line, the values of z it maps to start at 0 and extend one step at a time in both directions along the number line, so the function will cover all integers, each exactly once.