Discrete Structures

CSE 2315 (Spring 2014)

Lecture 2 Propositional Logic

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Smullyan's Island Puzzle

- You meet two inhabitants of Smullyan's Island (where each one is either a liar or a truth-teller).
  - A says, "Either B is lying or I am"
  - B says, "A is lying"
- Who is telling the truth ?

Introduction: Logic?

- We will study
  - Formal Logic
  - Propositional Logic (PL)
- Logic
  - is the study of the logic relationships between objects and
  - forms the basis of all mathematical reasoning and all automated reasoning

Introduction: PL?

- In Propositional Logic (a.k.a Propositional Calculus or Sentential Logic), the objects are called propositions.
- **Definition**: A proposition is a statement that is either true or false, but not both.
- We usually denote a proposition by a letter: *A*, *B*, *C*, *D*, …

Introduction: Proposition

- **Definition**: The value of a proposition is called its truth value; denoted by
  - T or 1 if it is true or
  - F or 0 if it is false
- Opinions, interrogative, and imperative are not propositions
- **Truth table**

A
T
F

Propositions: Examples

- The following are propositions
  - Today is Monday
  - The grass is wet
  - It is raining
- The following are not propositions
  - C++ is the best language *Opinion*
  - When is the pretest? *Interrogative*
  - Do your homework *Imperative*

Are these propositions?

- 2+2=5
- Every integer is divisible by 12
- She is very talented
- Ten is less than seven
- There are life forms on other planets in the universe
- $x^2 = 16$

Logical connectives

- Connectives are used to create a compound proposition from two or more propositions
  - Negation (denote $'$ or $\neg$)
  - And or logical conjunction (denoted $\wedge$)
  - Or or logical disjunction (denoted $\vee$)
  - XOR or exclusive or (denoted $\oplus$)
  - Implication (denoted $\Rightarrow$ or $\rightarrow$)
  - Equivalence (denoted $\Leftrightarrow$ or $\leftrightarrow$)

- We define the meaning (semantics) of the logical connectives using <u>truth tables</u>

Logical Connective: Negation

- A', the negation of a proposition A, is also a proposition
- Unary connective
- Examples:
  – Today is not Monday
  – It is not the case that today is Monday, etc.
- **Truth table**

A	*A'*
F	T
T	F

Logical Connective: Logical And

- The logical connective And is true only when both of the propositions are true. It is also called a <u>conjunction</u>

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- Examples
  - It is raining and it is warm
  - (2+3=5) and (1<2)
  - Schroedinger's cat is dead and Schroedinger's cat is not dead.
- Truth table

A	***B***	$A \wedge B$
F	F	
F	T	
T	F	
T	T	

Conjunction Truth Table

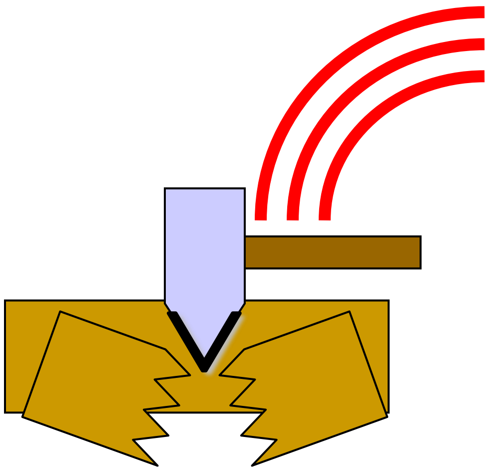
- Note that a conjunction $A_1 \wedge A_2 \wedge \ldots \wedge A_n$ of n propositions will have 2^n rows in its truth table.

A	B	$A \wedge B$
F	F	F
F	T	F
T	F	F
T	T	T

- $'$ and $\wedge$ operations together are universal, i.e., sufficient to express *any* truth table!

Logical Connective: Logical Or

- The logical disjunction, or logical Or, is true if one or both of the propositions are true.
- Examples
  - It is raining or it is the second lecture
  - $(2+2=5) \vee (1<2)$
  - You may have cake or ice cream
- **Truth table**



A	B	$A \wedge B$	$A \vee B$
F	F	F	
F	T	F	
T	F	F	
T	T	T	

Logical Connective: Exclusive Or

- The exclusive Or, or XOR, of two propositions is true when exactly one of the propositions is true and the other one is false
- Example
  - The circuit is either ON or OFF but not both
  - Let $ab<0$, then either $a<0$ or $b<0$ but not both
  - You may have cake or ice cream, but not both
- Truth table

A	B	$A \wedge B$	$A \vee B$	$A \oplus B$
F	F	F	F	
F	T	F	T	
T	F	F	T	
T	T	T	T	

A Simple Exercise

- Let
  A="It rained last night",
  B="The sprinklers came on last night,"
  C="The lawn was wet this morning."
- Translate each of the following into English:
- A' = "It didn't rain last night."
- $C \wedge A'$ = "The lawn was wet this morning, and it didn't rain last night."
- $C' \vee A \vee B$ = "Either the lawn wasn't wet this morning, or it rained last night, or the sprinklers came on last night."

Logical Connective: Implication (1)

- **Definition:** Let A and B be two propositions. The implication $A \rightarrow B$ is the proposition that is false when A is true and B is false and true otherwise
  - A is called the hypothesis, antecedent, premise
  - B is called the conclusion, consequence
- **Truth table**

A	B	$A \wedge B$	$A \vee B$	$A \oplus B$	$A \rightarrow B$
F	F	F	F	F	
F	T	F	T	T	
T	F	F	T	T	
T	T	T	T	F	

Logical Connective: Implication (2)

- The implication of $A \rightarrow B$ can be also read as
  - If A then B
  - A implies B
  - If A, B
  - A only if B
  - B if A
  - B when A
  - B whenever A
  - B follows from A
  - A is a sufficient condition for B (A is sufficient for B)
  - B is a necessary condition for A (B is necessary for A)

Logical Connective: Implication (3)

- "If I am elected, then I will lower the taxes next year".
- Under what conditions, the promise is broken, i.e., the statement is false ?
  - When I am elected, but I do not lower the taxes next year.
- For all other scenarios, I keep my promise, i.e., the statement is true.
  - I am elected, and lower the taxes next year
  - I am not elected, I lower the taxes next year.
  - I am not elected, I do not lower the taxes next year.

Logical Connective: Implication (4)

- Examples
  - If you buy your air ticket in advance, it is cheaper.
  - If x is an integer, then $x^2 \geq 0$.
  - If it rains, the grass gets wet.
  - If the sprinklers operate, the grass gets wet.
  - If 2+2=5, then all unicorns are pink.

Logical Connective: Implication (5)

- In spoken language, there are many ways to express implication (if … then…)
  - It rains, therefore the ground is wet.
    - Wet ground follows rain.
    - As long as it rains, the ground is wet.
    - Rain is a sufficient condition for the ground to be wet.
- When translating English to proposition forms
  - Rephrase sentence to "if …. Then…" without change its meaning

Exercise: Which of the following implications is true?

- If -1 is a positive number, then 2+2=5

  True. The premise is obviously false, thus no matter what the conclusion is, the implication holds.

- If -1 is a positive number, then 2+2=4

  True. Same as above.

- If $sin\ x = 0$, then $x = 0$

  False. x can be a multiple of π. If we let $x=2\pi$, then $sin\ x=0$ but $x\neq0$. The implication "if $sin\ x = 0$, then $x = k\pi$, for some k" is true.

Logical Connective: Equivalence (1)

- **Definition:** The equivalence $A \leftrightarrow B$ is the proposition that is true when A and B have the same truth values. It is false otherwise.
- Note that it is equivalent to $(A \rightarrow B) \wedge (B \rightarrow A)$
- **Truth table**

A	B	$A \wedge B$	$A \vee B$	$A \oplus B$	$A \rightarrow B$	$A \leftrightarrow B$
F	F	F	F	F	T	
F	T	F	T	T	T	
T	F	F	T	T	F	
T	T	T	T	F	T	

Logical Connective: Equivalence (2)

- The equivalence $A \leftrightarrow B$ can be equivalently read as
  - A if and only if B
  - A is a necessary and sufficient condition for B
  - if A then B, and conversely
  - A iff B
- Examples
  - $x>0$ if and only if x^2 is positive
  - The alarm goes off iff a burglar breaks in
  - You may have pudding iff you eat your meat

Exercise: Which of the following equivalence is true?

- $x^2 + y^2 = 0$ if and only if $x=0$ and $y=0$

  True. Both implications hold

- $2 + 2 = 4$ if and only if $\sqrt{2}<2$

  True. Both implications hold.

- $x^2 \geq 0$ if and only if $x \geq 0$

  False. The implication "if $x \geq 0$ then $x^2 \geq 0$" holds.
  However, the implication "if $x^2 \geq 0$ then $x \geq 0$" is false.
  Consider $x=-1$.
  The hypothesis $(-1)^2=1 \geq 0$ but the conclusion fails.