Discrete Structures

CSE 2315 (Spring 2014)

Lecture 3 Formal Logic

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Introduction: PL?

- In Propositional Logic (a.k.a Propositional Calculus or Sentential Logic), the objects are called propositions.
- **Definition**: A proposition is a statement that is either true or false, but not both.
- We usually denote a proposition by a letter: A, B, C, D, …
- E.g. There are life forms on other planets in the universe

Logical connectives

- Connectives are used to create a compound proposition from two or more propositions
  - Negation (denote $'$ or $\neg$)
  - And or logical conjunction (denoted $\wedge$)
  - Or or logical disjunction (denoted $\vee$)
  - XOR or exclusive or (denoted $\oplus$)
  - Implication (denoted $\Rightarrow$ or $\rightarrow$)
  - Equivalence (denoted $\Leftrightarrow$ or $\leftrightarrow$)

- We define the meaning (semantics) of the logical connectives using <u>truth tables</u>

Examples of Implications

- "If this lecture ever ends, then the sun will rise tomorrow." *True* or *False*?
- "If Tuesday is a day of the week, then I am a penguin." *True* or *False*?
- "If 1+1=6, then Obama is president." *True* or *False*?
- "If the moon is made of green cheese, then I am richer than Bill Gates." *True* or *False*?

Logical Connective: Equivalence (1)

- **Definition:** The equivalence $A \leftrightarrow B$ is the proposition that is true when A and B have the same truth values. It is false otherwise.
- Note that it is equivalent to $(A \rightarrow B) \wedge (B \rightarrow A)$
- **Truth table**

A	B	$A \wedge B$	$A \vee B$	$A \oplus B$	$A \rightarrow B$	$A \leftrightarrow B$
F	F	F	F	F	T	
F	T	F	T	T	T	
T	F	F	T	T	F	
T	T	T	T	F	T	

Logical Connective: Equivalence (2)

- The equivalence $A \leftrightarrow B$ can be equivalently read as
  - A if and only if B
  - A is a necessary and sufficient condition for B
  - if A then B, and conversely
  - A iff B
- Examples
  - $x>0$ if and only if x^2 is positive
  - The alarm goes off iff a burglar breaks in
  - You may have pudding iff you eat your meat

Exercise: Which of the following equivalence is true?

- $x^2 + y^2 = 0$ if and only if x=0 and y=0

  True. Both implications hold

- 2 + 2 = 4 if and only if $\sqrt{2}<2$

  True. Both implications hold.

- $x^2 \geq 0$ if and only if $x \geq 0$

  False. The implication "if $x \geq 0$ then $x^2 \geq 0$" holds.
  However, the implication "if $x^2 \geq 0$ then $x \geq 0$" is false.
  Consider x=-1.
  The hypothesis $(-1)^2=1 \geq 0$ but the conclusion fails.

Converse, Inverse, Contrapositive

- Consider the proposition $A \to B$
  - Its <u>converse</u> is the proposition $B \to A$
  - Its <u>inverse</u> is the proposition $A' \to B'$
  - Its <u>contrapositive</u> is the proposition $B' \to A'$

Truth Tables

- Truth tables are used to show/define the relationships between the truth values of
  - the individual propositions and
  - the compound propositions based on them

A	B	$A \wedge B$	$A \vee B$	$A \oplus B$	$A \rightarrow B$	$A \leftrightarrow B$
F	F	F	F	F	T	T
F	T	F	T	T	T	F
T	F	F	T	T	F	F
T	T	T	T	F	T	T

Constructing truth tables

- Construct the truth table for the following compound proposition

$$((A \wedge B) \vee B')$$

A	B	$A \wedge B$	B'	$((A \wedge B) \vee B')$
F	F	F	T	T
F	T	F	F	F
T	F	F	T	T
T	T	T	F	T

Well-formed Formulas

- Definition
  - A simple proposition is a well-formed formula (wff).
  - If A is a wff, then so is A′.
  - If A and B are wffs, then so are (A), $A \vee B$, $A \wedge B$, $A \rightarrow B$ and $A \leftrightarrow B$.
  - These are the only wffs.
- Example
  - $A \vee)B$ is not a wff.
- Precedence

  Ambiguity is resolved using the following order of precedence.
  - parentheses.
  - negation.
  - conjunction, disjunction.
  - implication.
  - equivalence.

  Use brackets and forget about precedence!

Tautologies

- Definition

  A wff which is always **true** is called a tautology, while a wff which is always **false** is called a contradiction.

- Example

  $A \vee A'$

Tautologies

- Definition

  If A and B are two wffs, and $A \leftrightarrow B$ is a tautology, then A and B are said to be **equivalent wffs** (denoted by $A \Leftrightarrow B$) and can be substituted for one another.

- Example

  $(A \rightarrow B) \Leftrightarrow (B' \rightarrow A')$

- Tautology checking

  How do you check if a wff is a tautology? Truth-tables!

Proving Equivalence via Truth Tables

- *Ex.* Prove that $p \vee q \Leftrightarrow (p' \wedge q')'$.

p	q	$p \vee q$	p'	q'	$p' \wedge q'$	$(p' \wedge q')'$
F	F	F	T	T	T	F
F	T	T	T	F	F	T
T	F	T	F	T	F	T
T	T	T	F	F	F	T

Common Tautological Equivalences

- De Morgan's Laws

  $(A \vee B)' \Leftrightarrow (A' \wedge B')$ $(A \wedge B)' \Leftrightarrow (A' \vee B')$
- Commutativity

  $(A \vee B) \Leftrightarrow (B \vee A)$ $(A \wedge B) \Leftrightarrow (B \wedge A)$
- Associativity

  $(A \vee B) \vee C \Leftrightarrow A \vee (B \vee C)$ $(A \wedge B) \wedge C \Leftrightarrow A \wedge (B \wedge C)$
- Distributivity

  $A \vee (B \wedge C) \Leftrightarrow (A \vee B) \wedge (A \vee C)$ $A \wedge (B \vee C) \Leftrightarrow (A \wedge B) \vee (A \wedge C)$
- Exercise

  Prove all the above tautologies.

Homework and Slides Strategies

- Either type up or handwritten is ok. The handwritten work has to be clear and recognizable.
- How to hand in? Please choose one of the following ways:
  - email your homework to me at heng@uta.edu, or
  - hand it in class (preferred).
  - your homework should be turned in before or in the class on due date.
- Please put your name and ID number on your homework. There is no requirement on layout and font.