# Discrete Structures

## CSE 2315 (Spring 2014)

## Lecture 7 Predicate Logic Proof Sequence

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# Converting English to Predicate Logic

- Not an easy task! More than one result possible, depending on semantics of English language (which is not unambiguous). "Hang him not, let him free" and "Hang him, not let him free"!
- Example
  - All parrots are ugly.
  - Some parrots are ugly.
  - All dogs chase all rabbits.
  - Some dogs chase all rabbits.

# Exercise

- Let S(x) denote "x is a student", I(x) denote "x is intelligent" and M(x) denote "x likes music". Give predicate wffs for:
  - All students are intelligent.
  - Some intelligent students like music.
  - Only intelligent students like music.

# Validity

- A predicate wff is valid, if it is true in all interpretations. Validity in Predicate logic plays the role of tautology in propositional logic.

- Four fundamentally valid predicate wffs

(i) $(\forall x)P(x) \rightarrow (\exists x)P(x)$.

(ii) $(\forall x)P(x) \rightarrow P(a)$.

(iii) $(\forall x)(P(x) \wedge Q(x)) \rightarrow [(\forall x)P(x) \wedge (\forall x)Q(x)]$.

(iv) $P(x) \rightarrow [Q(x) \rightarrow P(x)]$.

# Motivation

- From “All humans are mortal”, and “Socrates is human”, we wish to conclude that “Socrates is mortal.” We therefore need rules to reason about predicate expressions. Symbolically,

$$[(\forall x)(H(x) \to M(x)) \land H(s)] \to M(s)$$

- All the rules of propositional logic work; however, two points need to be made:
  - A single atom in a predicate expression includes the quantifier. For instance,

    $$[((\forall x)P(x) \to (\forall x)Q(x)) \land (\forall x)P(x)] \to (\forall x)Q(x)$$

    is a valid argument in predicate logic.
  - Propositional rules are not sufficient. For instance, you cannot use propositional rules to conclude validity in the Socrates example.

# Universal Instantiation

- From $(\forall x)P(x)$, you can conclude $P(t)$, where t is any constant or variable.
- Rule is abbreviated as u.i.
- If t is a variable, it must not fall within the scope of a quantifier for t. For instance, from $(\forall x)(\exists y)P(x, y)$, you cannot conclude $(\exists y)P(y, y)$. (Domain of integers).

- Let us prove that the following argument is valid, using ui.

$$[(\forall x)[H(x) \to M(x)] \wedge H(s)] \to M(s)$$

# Existential Instantiation

- From $(\exists x)P(x)$, you can conclude $P(a)$, where a is a constant symbol not used previously in the proof sequence.
- Rule is abbreviated as e.i.
- Must be the first rule that introduces a.

- Example

Show that $[(\forall x)[P(x) \to Q(x)] \wedge (\exists y)P(y)] \to (\exists y)Q(y)$ is valid.

# Universal Generalization

- From P(x), you can conclude $(\forall x)P(x)$.
- Rule is abbreviated as u.g.
- P(x) has not been deduced from a hypothesis in which x is a free variable. Also, P(x) has not been deduced using e.i.

- Example

  Show that the following argument is valid.

$$[(\forall x)[P(x) \rightarrow Q(x)] \wedge (\forall x)P(x)] \rightarrow (\forall x)Q(x)$$

# Existential Generalization

- From P(x) or P(a), you can conclude (∃x)P(x).
- Rule is abbreviated as e.g.
- To go from P(a) to (∃x)P(x), x must not appear in P(a). Otherwise, we could derive (∃y)P(y, y) from P(a, y)! The argument P(a, y) → (∃y)P(y, y) is simply not valid. (Why?)

- Some more examples

$$(\forall x)[P(x) \wedge Q(x)] \rightarrow (\forall x)P(x) \wedge (\forall x)Q(x).$$
$$[(\forall y)[P(x) \rightarrow Q(x, y)]] \rightarrow [P(x) \rightarrow (\forall y)Q(x, y)].$$
$$[P(x) \rightarrow (\forall y)Q(x, y)] \rightarrow (\forall y)[P(x) \rightarrow Q(x, y)].$$

# Two Laws

- Negation
  - $[(\exists x)A(x)]' \Leftrightarrow (\forall x)[A(x)]'$
  - $[(\forall x)A(x)]' \Leftrightarrow (\exists x)[A(x)]'$

- The proof of the negation rules are tedious and I do not expect you to know them. Feel free to use the above two rules as and when you need them, without proof.

- One more example

  Establish the validity of the following statement:

$$(\forall x)[P(x) \vee Q(x)] \rightarrow [(\exists x)P(x) \vee (\forall x)Q(x)]$$

# A Verbal Argument

- Every microcomputer has a serial interface port. Some microcomputers have a parallel port. Therefore, some microcomputers have both a serial and a parallel port. Symbolically,

$$[(\forall x)[M(x) \rightarrow S(x)] \wedge (\exists x)[M(x) \wedge P(x)]] \rightarrow (\exists x)[M(x) \wedge S(x) \wedge P(x)]$$