# Discrete Structures

## CSE 2315 (Spring 2014)

## Lecture 9 Proof

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# Proof (Chapter 2)

- Arguments are statements having the form P → Q or more generally P(x) →Q(x). We are no longer interested in establishing validity, i.e., truth in all interpretations. In Mathematics, we are interested in establishing **truth** in a specific interpretation. For instance, Pythagoras' theorem applies to the domain of triangles, Fermat's theorem applies to numbers and so on.
- Arguments which are contextually true (as opposed to being universally true) are called **theorems**. If an argument is not yet proven, it is called a conjecture.
- How to prove theorems? Simply add additional facts as hypotheses and then use rules of predicate logic (or propositional logic)!

# Proof Techniques

- Applicable to all domains
  - Exhaustive proof.
  - Direct proof.
  - Proof by contraposition.
  - Proof by contradiction.
  - Serendipity.

- Applicable to Structured Domains
  - Mathematical Induction.
  - Diagonalization.

# Exhaustive Proof

- Simply enumerate all elements of the domain and check if the argument holds for each element. If it does, then the conjecture is a theorem.

  Let D = {1, 2, 3, 4, 5, 6, 7, 8}. If $x \in D$ and x is divisible by 4, then x is divisible by 2.

  Let $P(x) \equiv$ x is divisible by 4, and $Q(x) \equiv$ x is divisible by 2. The conjecture is $(\forall x \in D) P(x) \rightarrow Q(x)$.

  Check for x = 1, x = 2, . . . . ...
- Only works when the domain is finite.
- Example

Let $D = \{0, 1, 2, 3, 4, 5\}$. For all $x \in D$, $x^2 \leq 10 + 5 \cdot x$.

# Direct Proof

- Given the conjecture P → Q, assume that P is true and show that Q must be true.

- Example

  Show that the product of two even integers is even.

  Formally, $(\forall x)(\forall y)$ (x is even $\wedge$ y is even) → xy is even

- Proof.

  Since x is even, $x = 2 \cdot k$, for some integer k.

  Since y is even, $y = 2 \cdot r$ , for some integer r .

  Therefore, $xy = (2 \cdot k) \cdot (2 \cdot r) = 2 \cdot (2 \cdot k \cdot r) = 2 \cdot p$, for some integer p.

  It follows that xy is even.

- Example

  Show that if an integer is divisible by 4, then it is divisible by 2.

# Proof by Contraposition

- Given the argument $P \rightarrow Q$, use the direct proof technique to show that $Q' \rightarrow P'$. We know that $P \rightarrow Q \Leftrightarrow Q' \rightarrow P'$!

- Do not confuse contrapositive with converse!

- Example

  Show that if the square of an integer is odd, then x must be odd. Formally, ($x^2$ is odd) $\rightarrow$ x is odd

  Proof.

  We will instead show that if x is even, then $x^2$ must be even! But this follows from the example used in Direct Proof!

# Proof by Contradiction

- Show that $[(P \wedge Q') \rightarrow \textbf{false}] \rightarrow (P \rightarrow Q)$ is a tautology. It follows that if we show that $P \wedge Q'$ is unequivocally **false**, we have in fact, proven $P \rightarrow Q$!

- Definition A rational number is one that can be expressed in the form p/q , where p and q are integers, with no common divisor and q != 0. The former condition is denoted by gcd(p, q) = 1.

- Example

  Show that $\sqrt{2}$ is not rational.

# Serendipity

- Prayer! Good Luck.

- Example

  Number of games in a tennis tournament.