Machine Learning

CSE 6363 (Fall 2016)

Lecture 6 Naïve Bayes and Linear Regression

Heng Huang, Ph.D.

Department of Computer Science and Engineering

Classification

- **Learn**: h:**X** $\mapsto$ Y
  - **X** – features
  - Y – target classes

- Suppose you know P(Y|**X**) exactly, how should you classify?
  - Bayes classifier:

$$P(Y|X) = \frac{P(X|Y)P(Y)}{P(X)}$$

Which is shorthand for:

$$(\forall i, j) P(Y = y_i | X = x_j) = \frac{P(X = x_j | Y = y_i) P(Y = y_i)}{P(X = x_j)}$$

Ref: Carlos Guestrin

How to Learn the Classifier?

Sky	Temp	Humid	Wind	Water	Forecst	EnjoySpt
Sunny	Warm	Normal	Strong	Warm	Same	Yes
Sunny	Warm	High	Strong	Warm	Same	Yes
Rainy	Cold	High	Strong	Warm	Change	No
Sunny	Warm	High	Strong	Cool	Change	Yes

- How do we represent these? How many parameters?
  - Prior, P(Y):
    - Suppose Y is composed of *k* classes
  - Likelihood, P(**X**|Y):
    - Suppose **X** is composed of *n* binary features

Ref: Carlos Guestrin

The Naïve Bayes Assumption

- Naïve Bayes assumption:
  - Features are independent given class:

$$P(X_1, X_2|Y) = P(X_1|X_2, Y)P(X_2|Y) \\ = P(X_1|Y)P(X_2|Y)$$

  - More generally:

$$\boxed{P(X_1 \ldots X_n|Y) = \prod_i P(X_i|Y)}$$

- How many parameters now?
  - Suppose **X** is composed of *n* binary features

Ref: Carlos Guestrin

The Naïve Bayes Classifier

- Given:
  - Prior P(Y)
  - *n* conditionally independent features **X** given the class Y
  - For each X_i, we have likelihood $P(X_i|Y)$
- Decision rule:

$$y^* = h_{NB}(\mathbf{x}) = \arg\max_y P(y)P(x_1, \ldots, x_n \mid y)$$
$$= \arg\max_y P(y) \prod_i P(x_i|y)$$

Ref: Carlos Guestrin

Text Classification

- Classify e-mails
  - Y = {Spam,NotSpam}
- Classify news articles
  - Y = {what is the topic of the article?}
- Classify webpages
  - Y = {Student, professor, project, …}

- What about the features **X**?
  - The text!

Ref: Carlos Guestrin

Features X Are Entire Document

Article from rec.sport.hockey

Path: cantaloupe.srv.cs.cmu.edu!das-news.harvard.e
From: xxx@yyy.zzz.edu (John Doe)
Subject: Re: This year's biggest and worst (opinic
Date: 5 Apr 93 09:53:39 GMT

I can only comment on the Kings, but the most
obvious candidate for pleasant surprise is Alex
Zhitnik. He came highly touted as a defensive
defenseman, but he's clearly much more than that.
Great skater and hard shot (though wish he were
more accurate). In fact, he pretty much allowed
the Kings to trade away that huge defensive
liability Paul Coffey. Kelly Hrudey is only the
biggest disappointment if you thought he was any
good to begin with. But, at best, he's only a
mediocre goaltender. A better choice would be
Tomas Sandstrom, though not through any fault of
his own, but because some thugs in Toronto decidec

Ref: Carlos Guestrin

NB for Text Classification

- $P(\mathbf{X}|Y)$ is huge!!!
  - Article at least 1000 words, $\mathbf{X}=\{X_1,\ldots,X_{1000}\}$
  - X_i represents i^{th} word in document, i.e., the domain of X_i is entire vocabulary, e.g., Webster Dictionary (or more), 10,000 words, etc.

- NB assumption helps a lot!!!
  - $P(X_i=x_i|Y=y)$ is just the probability of observing word x_i in a document on topic y

$$h_{NB}(\mathbf{x}) = \arg\max_y P(y) \prod_{i=1}^{LengthDoc} P(x_i|y)$$

Ref: Carlos Guestrin

Bag of Words Model

- Typical additional assumption – **Position in document doesn't matter**: $P(X_i=x_i|Y=y) = P(X_k=x_i|Y=y)$
  - "Bag of words" model – order of words on the page ignored
  - Sounds really silly, but often works very well!

$$P(y) \prod_{i=1}^{LengthDoc} P(x_i|y)$$

aardvark	0
about	2
all	2
Africa	1
apple	0
anxious	0
...	
gas	1
...	
oil	1
…	
Zaire	0

NB with Bag of Words for Text Classification

- Learning phase:
  - Prior P(Y)
    - Count how many documents you have from each topic (+ prior)
  - $P(X_i|Y)$
    - For each topic, count how many times you saw word in documents of this topic (+ prior)
- Test phase:
  - For each document
    - Use naïve Bayes decision rule

$$h_{NB}(\mathbf{x}) = \arg\max_y P(y) \prod_{i=1}^{LengthDoc} P(x_i|y)$$

Ref: Carlos Guestrin

Twenty News Groups Results

Given 1000 training documents from each group
Learn to classify new documents according to
which newsgroup it came from

comp.graphics	misc.forsale
comp.os.ms-windows.misc	rec.autos
comp.sys.ibm.pc.hardware	rec.motorcycles
comp.sys.mac.hardware	rec.sport.baseball
comp.windows.x	rec.sport.hockey
alt.atheism	sci.space
soc.religion.christian	sci.crypt
talk.religion.misc	sci.electronics
talk.politics.mideast	sci.med
talk.politics.misc	
talk.politics.guns	

Naive Bayes: 89% classification accuracy

Ref: Carlos Guestrin

Supervised Learning

Data: $D = \{D_1, D_2, .., D_n\}$ **a set of *n* examples**

$D_i = < \mathbf{x}_i, y_i >$

$\mathbf{x}_i = (x_{i,1}, x_{i,2}, \cdots x_{i,d})$ is an input vector of size d

y_i is the desired output (given by a teacher)

Objective: learn the mapping $f : X \to Y$

s.t. $y_i \approx f(\mathbf{x}_i)$ for all $i = 1,.., n$

- **Regression:** Y is **continuous**

  Example: earnings, product orders $\longrightarrow$ company stock price
- **Classification:** Y is **discrete**

  Example: handwritten digit in binary form $\longrightarrow$ digit label

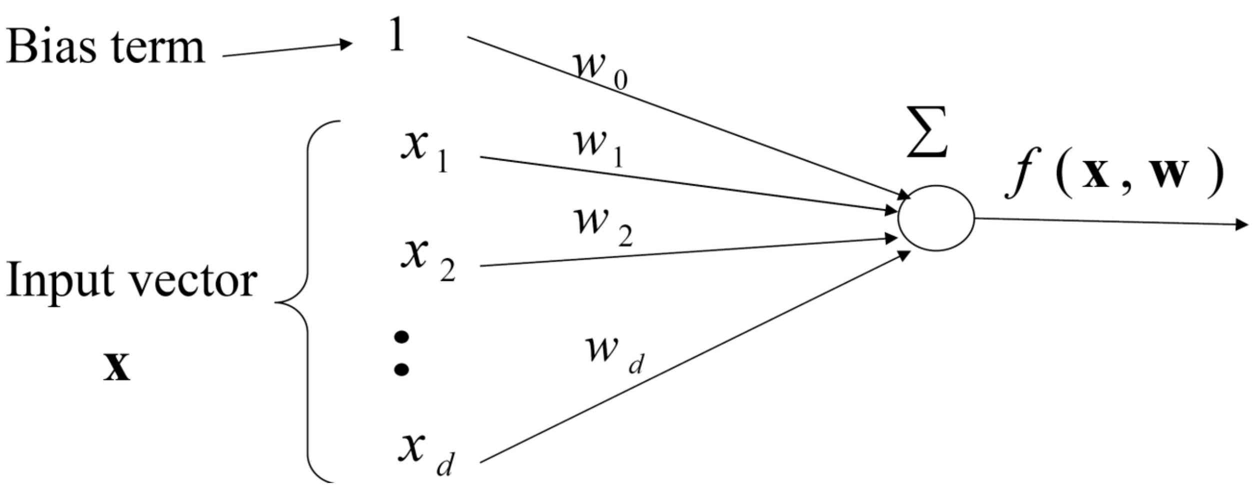
Ref: Milos Hauskrecht

Linear Regression

- **Function** $f : X \to Y$ is a linear combination of input components

$$f(\mathbf{x}) = w_0 + w_1 x_1 + w_2 x_2 + \ldots w_d x_d = w_0 + \sum_{j=1}^{d} w_j x_j$$

$w_0, w_1, \ldots w_k$ **- parameters (weights)**



Ref: Milos Hauskrecht

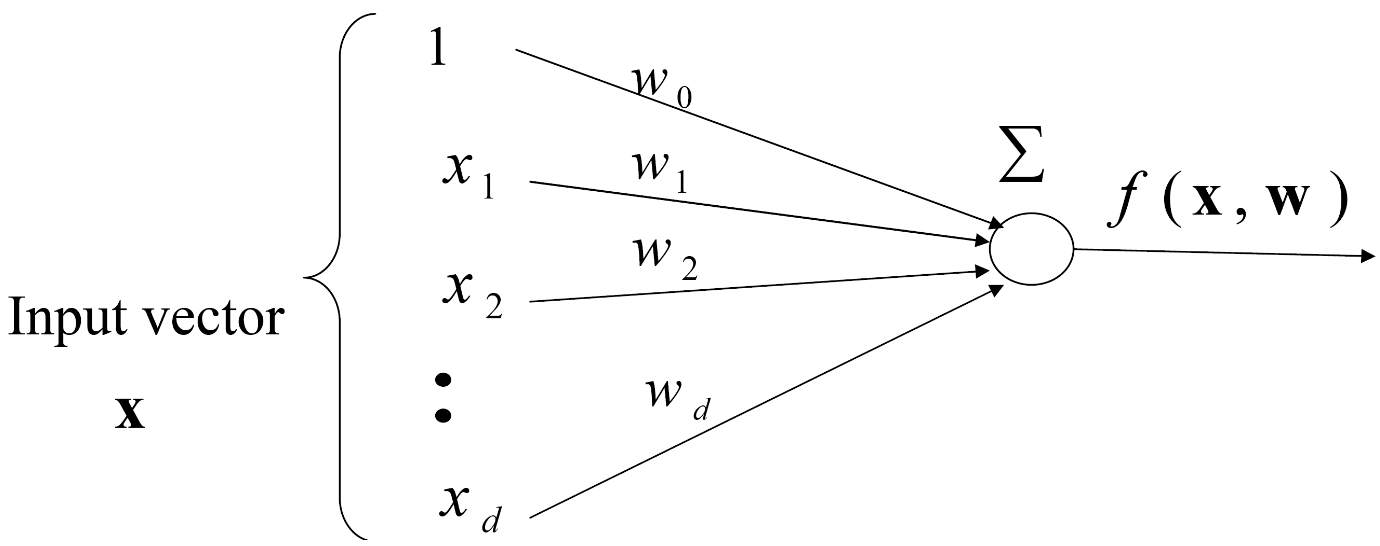
Linear Regression

- **Shorter (vector) definition of the model**
  - Include bias constant in the input vector

$$\mathbf{x} = (1, x_1, x_2, \cdots x_d)$$

$$f(\mathbf{x}) = w_0 x_0 + w_1 x_1 + w_2 x_2 + \ldots w_d x_d = \mathbf{w}^T \mathbf{x}$$

$w_0, w_1, \ldots w_k$ **- parameters (weights)**

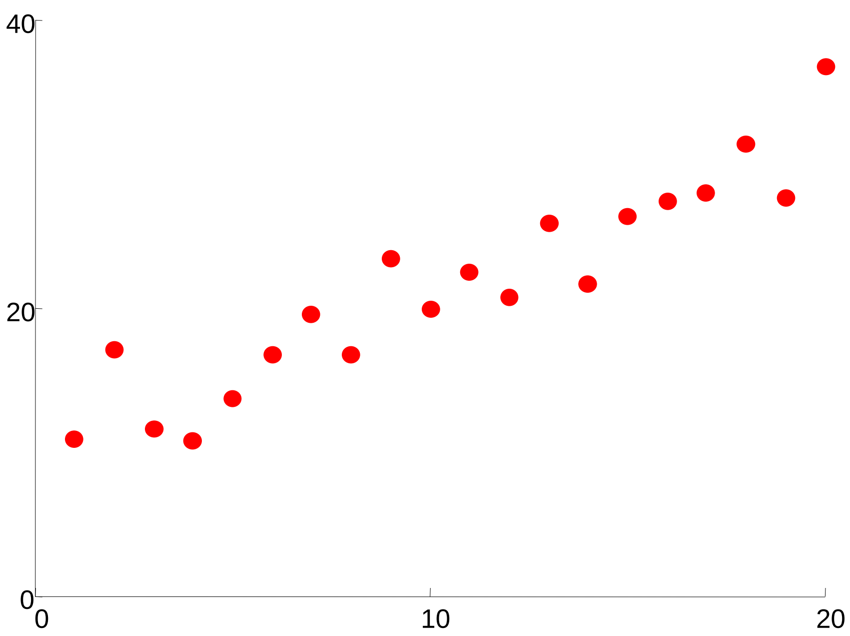


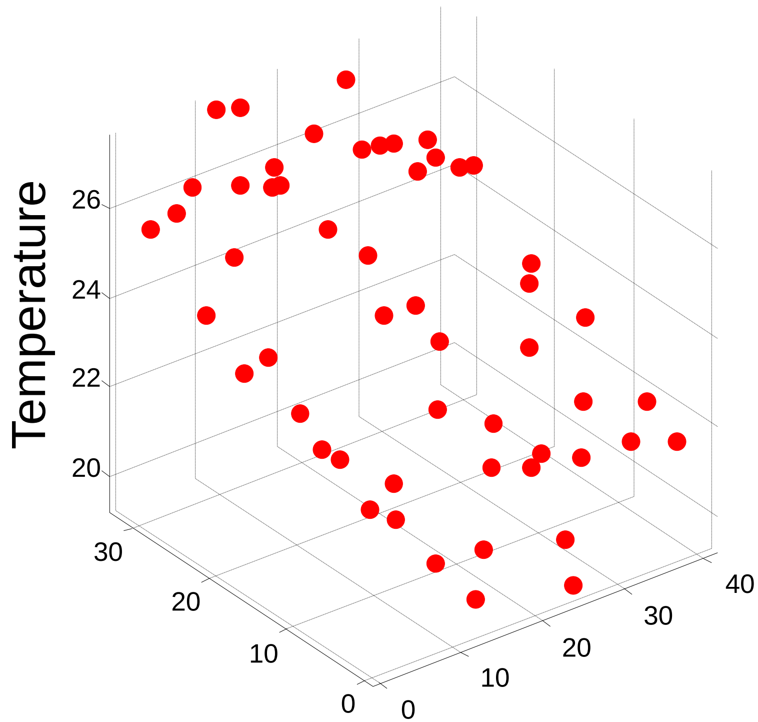
Ref: Milos Hauskrecht

Examples

- Voltage -> Temperature
- Stock prediction -> Money
- Processes, memory -> Power consumption
- Protein structure -> Energy
- Robot arm controls -> Torque at effector
- Location, industry, past losses -> Premium

Linear Regression

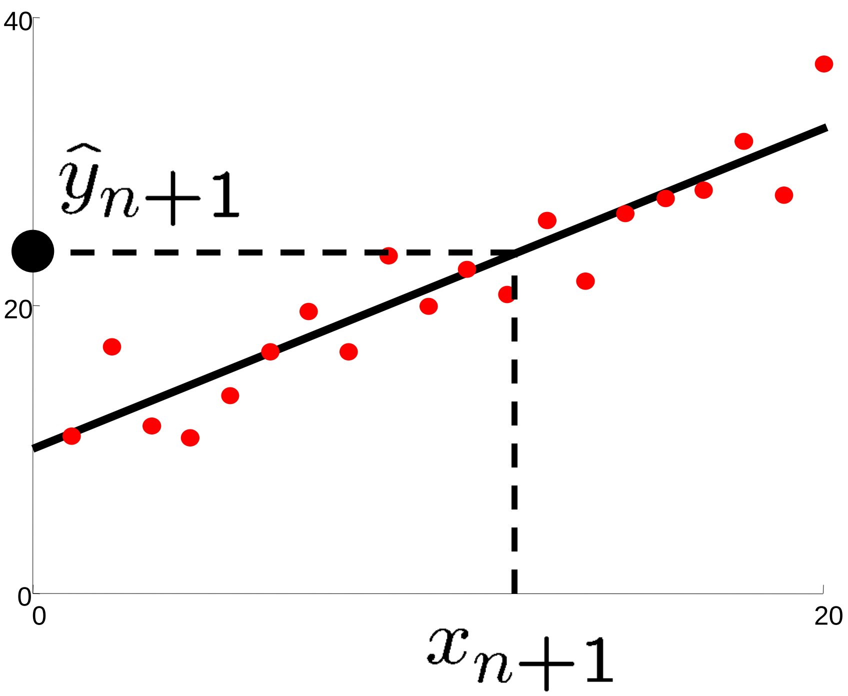


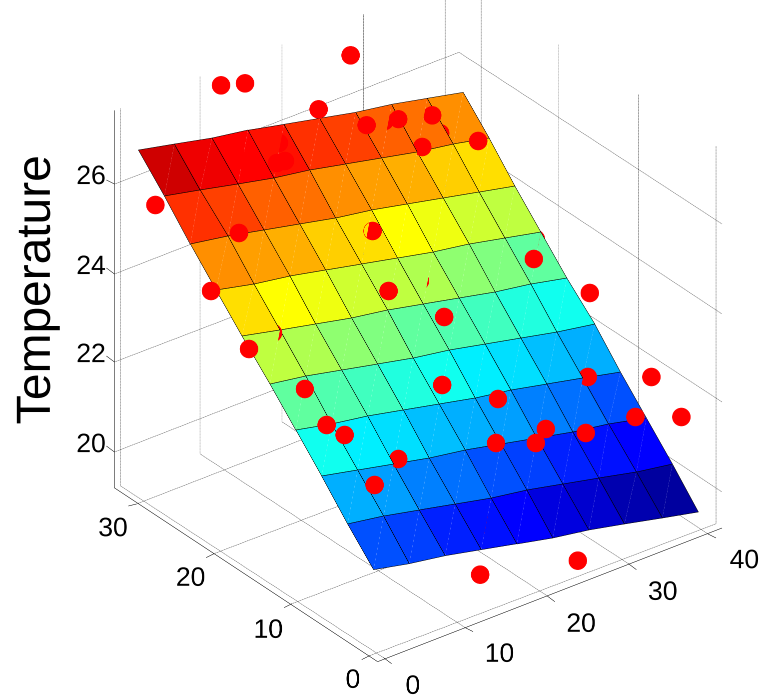


Given examples $(x_i, y_i)_{i=1\ldots n}$

Predict y_{n+1} given a new point x_{n+1}

Linear Regression





Prediction

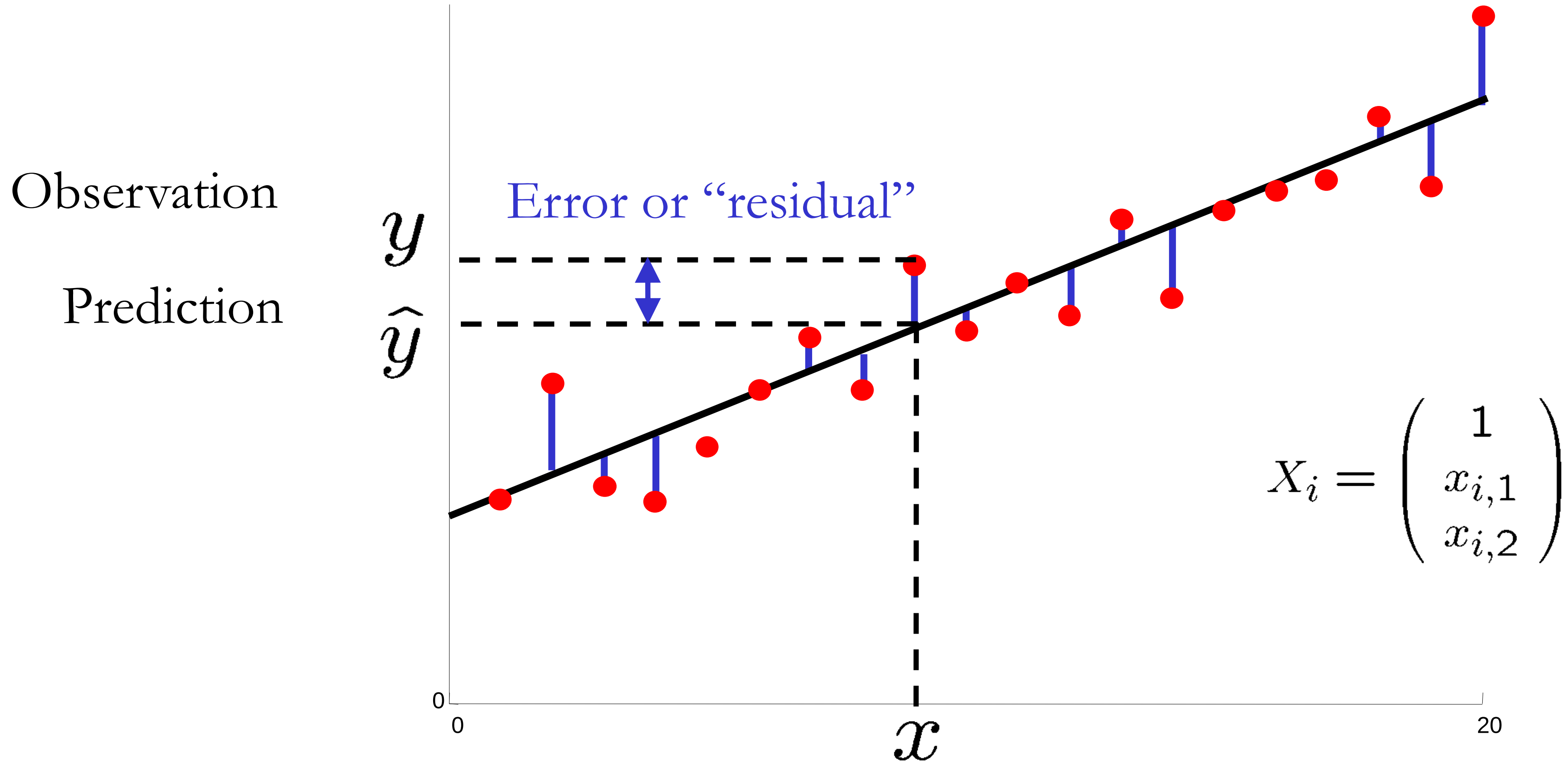
$$\hat{y}_i = w_0 + w_1 x_i$$

Pre

$$\hat{y}_i = \begin{pmatrix} 1 & x_{i,1} & x_{i,2} \end{pmatrix} \begin{pmatrix} w_0 \\ w_1 \\ w_2 \end{pmatrix} x_{i,2}$$

$$= X_i^\top w$$

Ordinary Least Squares (OLS)



Sum squared error $\sum_i (X_i^\top w - y_i)^2$

Linear Regression. Optimization.

- We want the **weights minimizing the error**

$$J_n = \frac{1}{n}\sum_{i=1,..n}(y_i - f(\mathbf{x}_i))^2 = \frac{1}{n}\sum_{i=1,..n}(y_i - \mathbf{w}^T\mathbf{x}_i)^2$$

- For the optimal set of parameters, derivatives of the error with respect to each parameter must be 0

$$\frac{\partial}{\partial w_j} J_n(\mathbf{w}) = -\frac{2}{n}\sum_{i=1}^{n}(y_i - w_0 x_{i,0} - w_1 x_{i,1} - \ldots - w_d x_{i,d})x_{i,j} = 0$$

- **Vector of derivatives:**

$$\text{grad}_{\mathbf{w}}(J_n(\mathbf{w})) = \nabla_{\mathbf{w}}(J_n(\mathbf{w})) = -\frac{2}{n}\sum_{i=1}^{n}(y_i - \mathbf{w}^T\mathbf{x}_i)\mathbf{x}_i = \bar{\mathbf{0}}$$

Ref: Milos Hauskrecht

Linear Regression. Optimization.

- $\text{grad}_{\mathbf{w}}(J_n(\mathbf{w})) = \bar{\mathbf{0}}$ defines a set of equations in **w**

$$\frac{\partial}{\partial w_0} J_n(\mathbf{w}) = -\frac{2}{n}\sum_{i=1}^{n}(y_i - w_0 x_{i,0} - w_1 x_{i,1} - \ldots - w_d x_{i,d}) = 0$$

$$\frac{\partial}{\partial w_1} J_n(\mathbf{w}) = -\frac{2}{n}\sum_{i=1}^{n}(y_i - w_0 x_{i,0} - w_1 x_{i,1} - \ldots - w_d x_{i,d}) x_{i,1} = 0$$

$$\ldots$$

$$\frac{\partial}{\partial w_j} J_n(\mathbf{w}) = -\frac{2}{n}\sum_{i=1}^{n}(y_i - w_0 x_{i,0} - w_1 x_{i,1} - \ldots - w_d x_{i,d}) x_{i,j} = 0$$

$$\ldots$$

$$\frac{\partial}{\partial w_d} J_n(\mathbf{w}) = -\frac{2}{n}\sum_{i=1}^{n}(y_i - w_0 x_{i,0} - w_1 x_{i,1} - \ldots - w_d x_{i,d}) x_{i,d} = 0$$

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Solving Linear Regression

$$\frac{\partial}{\partial w_j} J_n(\mathbf{w}) = -\frac{2}{n}\sum_{i=1}^{n}(y_i - w_0 x_{i,0} - w_1 x_{i,1} - \ldots - w_d x_{i,d})x_{i,j} = 0$$

By rearranging the terms we get a **system of linear equations** with *d+1* unknowns

$$\boxed{\mathbf{Aw} = \mathbf{b}}$$

$$w_0 \sum_{i=1}^{n} x_{i,0} 1 + w_1 \sum_{i=1}^{n} x_{i,1} 1 + \ldots + w_j \sum_{i=1}^{n} x_{i,j} 1 + \ldots + w_d \sum_{i=1}^{n} x_{i,d} 1 = \sum_{i=1}^{n} y_i 1$$

$$w_0 \sum_{i=1}^{n} x_{i,0} x_{i,1} + w_1 \sum_{i=1}^{n} x_{i,1} x_{i,1} + \ldots + w_j \sum_{i=1}^{n} x_{i,j} x_{i,1} + \ldots + w_d \sum_{i=1}^{n} x_{i,d} x_{i,1} = \sum_{i=1}^{n} y_i x_{i,1}$$

•••

$$w_0 \sum_{i=1}^{n} x_{i,0} x_{i,j} + w_1 \sum_{i=1}^{n} x_{i,1} x_{i,j} + \ldots + w_j \sum_{i=1}^{n} x_{i,j} x_{i,j} + \ldots + w_d \sum_{i=1}^{n} x_{i,d} x_{i,j} = \sum_{i=1}^{n} y_i x_{i,j}$$

•••

Ref: Milos Hauskrecht

Solving Linear Regression

- The optimal set of weights satisfies:

$$\nabla_{\mathbf{w}}(J_n(\mathbf{w})) = -\frac{2}{n}\sum_{i=1}^{n}(y_i - \mathbf{w}^T\mathbf{x}_i)\mathbf{x}_i = \overline{\mathbf{0}}$$

Leads to a **system of linear equations (SLE)** with *d+1* unknowns of the form

$$\mathbf{A}\mathbf{w} = \mathbf{b}$$

$$w_0\sum_{i=1}^{n}x_{i,0}x_{i,j} + w_1\sum_{i=1}^{n}x_{i,1}x_{i,j} + \ldots + w_j\sum_{i=1}^{n}x_{i,j}x_{i,j} + \ldots + w_d\sum_{i=1}^{n}x_{i,d}x_{i,j} = \sum_{i=1}^{n}y_i x_{i,j}$$

Solution to SLE:

$$\boxed{\mathbf{w} = \mathbf{A}^{-1}\mathbf{b}}$$

Gradient Descent Solution

Goal: the weight optimization in the linear regression model

$$J_n = Error(\mathbf{w}) = \frac{1}{n} \sum_{i=1,..n} (y_i - f(\mathbf{x}_i, \mathbf{w}))^2$$

An alternative to SLE solution:

- **Gradient descent**

  Idea:

  – Adjust weights in the direction that improves the Error

  – The gradient tells us what is the right direction

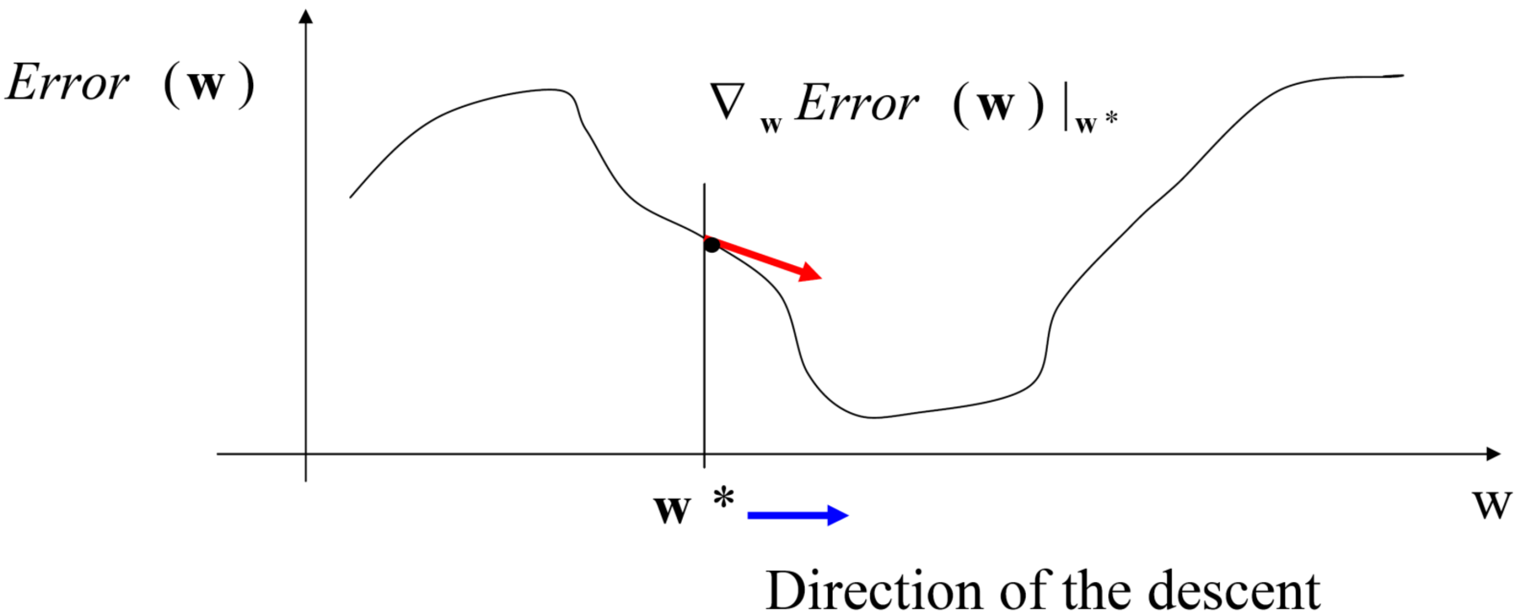
$$\mathbf{w} \leftarrow \mathbf{w} - \alpha \nabla_{\mathbf{w}} Error_i(\mathbf{w})$$

$\alpha > 0$ - a **learning rate** (scales the gradient changes)

Ref: Milos Hauskrecht

Gradient Descent Method

- Descend using the gradient information



- Change the value of $\mathbf{w}$ according to the gradient

$$\mathbf{w} \leftarrow \mathbf{w} - \alpha \nabla_{\mathbf{w}} Error_i(\mathbf{w})$$

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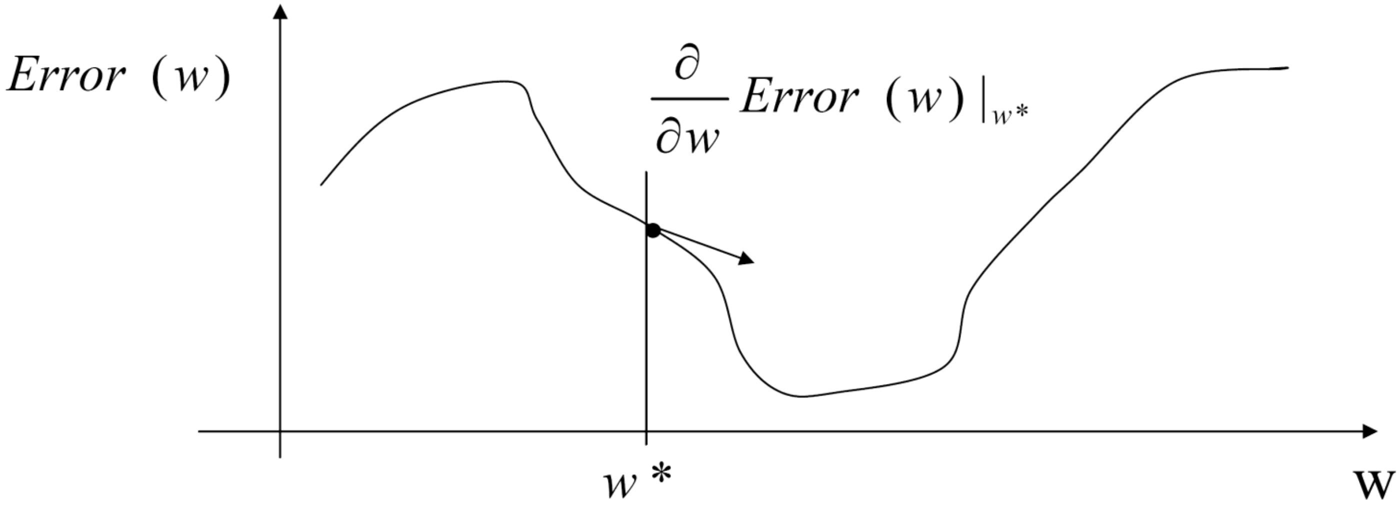
Gradient Descent Method



- New value of the parameter

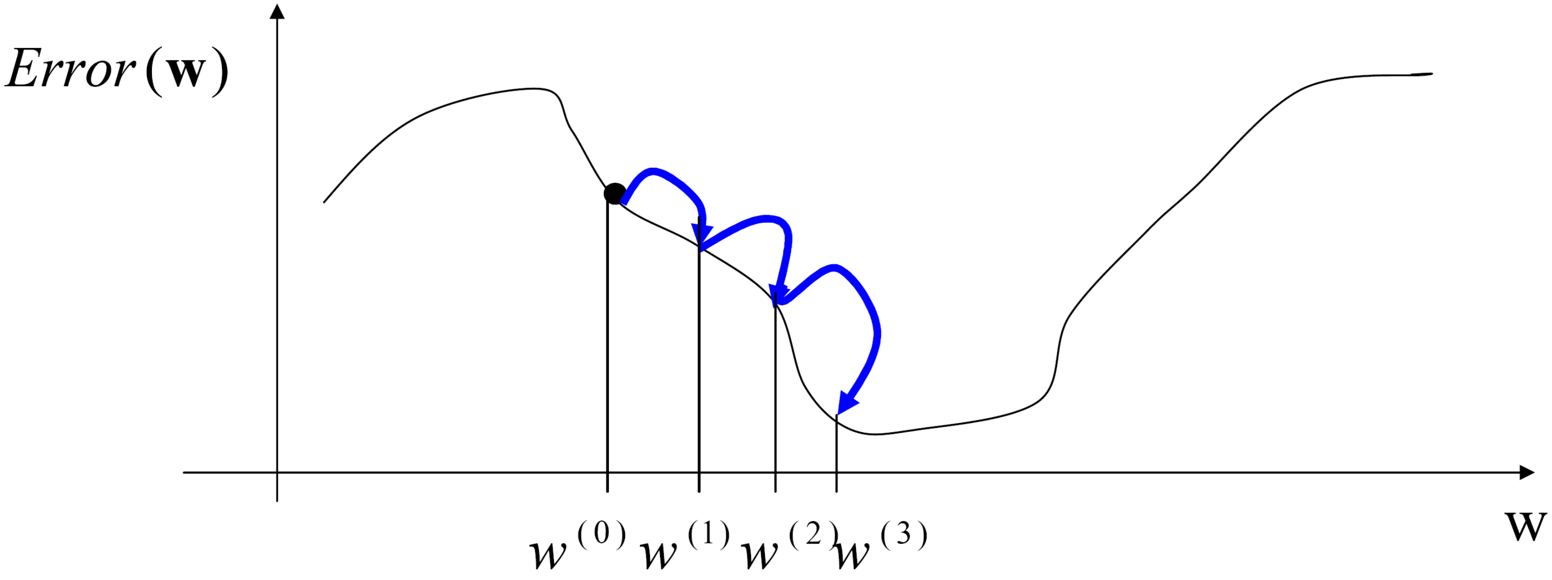
$$w_j \leftarrow w_j^* - \alpha \frac{\partial}{\partial w_j} Error\ (w)|_{w^*} \qquad \text{For all j}$$

$\alpha > 0$ - a learning rate (scales the gradient changes)

Ref: Milos Hauskrecht

Gradient Descent Method

- Iteratively approaches the optimum of the Error function



Ref: Milos Hauskrecht

Online Gradient Algorithm

Linear model $$f(\mathbf{x}) = \mathbf{w}^T \mathbf{x}$$

On-line error $$J_{online} = Error_i(\mathbf{w}) = \frac{1}{2}(y_i - f(\mathbf{x}_i, \mathbf{w}))^2$$

On-line algorithm: generates a sequence of online updates

(i)-th update step with : $D_i =< \mathbf{x}_i, y_i >$

j-th weight:

$$w_j^{(i)} \leftarrow w_j^{(i-1)} - \alpha(i) \frac{\partial Error_i(\mathbf{w})}{\partial w_j} \Big|_{\mathbf{w}^{(i-1)}}$$

$$\boxed{w_j^{(i)} \leftarrow w_j^{(i-1)} + \alpha(i)(y_i - f(\mathbf{x}_i, \mathbf{w}^{(i-1)}))x_{i,j}}$$

Fixed learning rate: $\alpha(i) = C$

- Use a small constant

Annealed learning rate: $\alpha(i) \approx \frac{1}{i}$

- Gradually rescales changes

Ref: Milos Hauskrecht

Online Regression Algorithm

Online-linear-regression (*D, number of iterations*)

Initialize weights $\mathbf{w} = (w_0, w_1, w_2 \dots w_d)$

for *i*=1:1*: number of iterations*

do **select** a data point $D_i = (\mathbf{x}_i, y_i)$ from *D*

set learning rate $\alpha(i)$

update weight vector

$$\mathbf{w} \leftarrow \mathbf{w} + \alpha(i)(y_i - f(\mathbf{x}_i, \mathbf{w}))\mathbf{x}_i$$

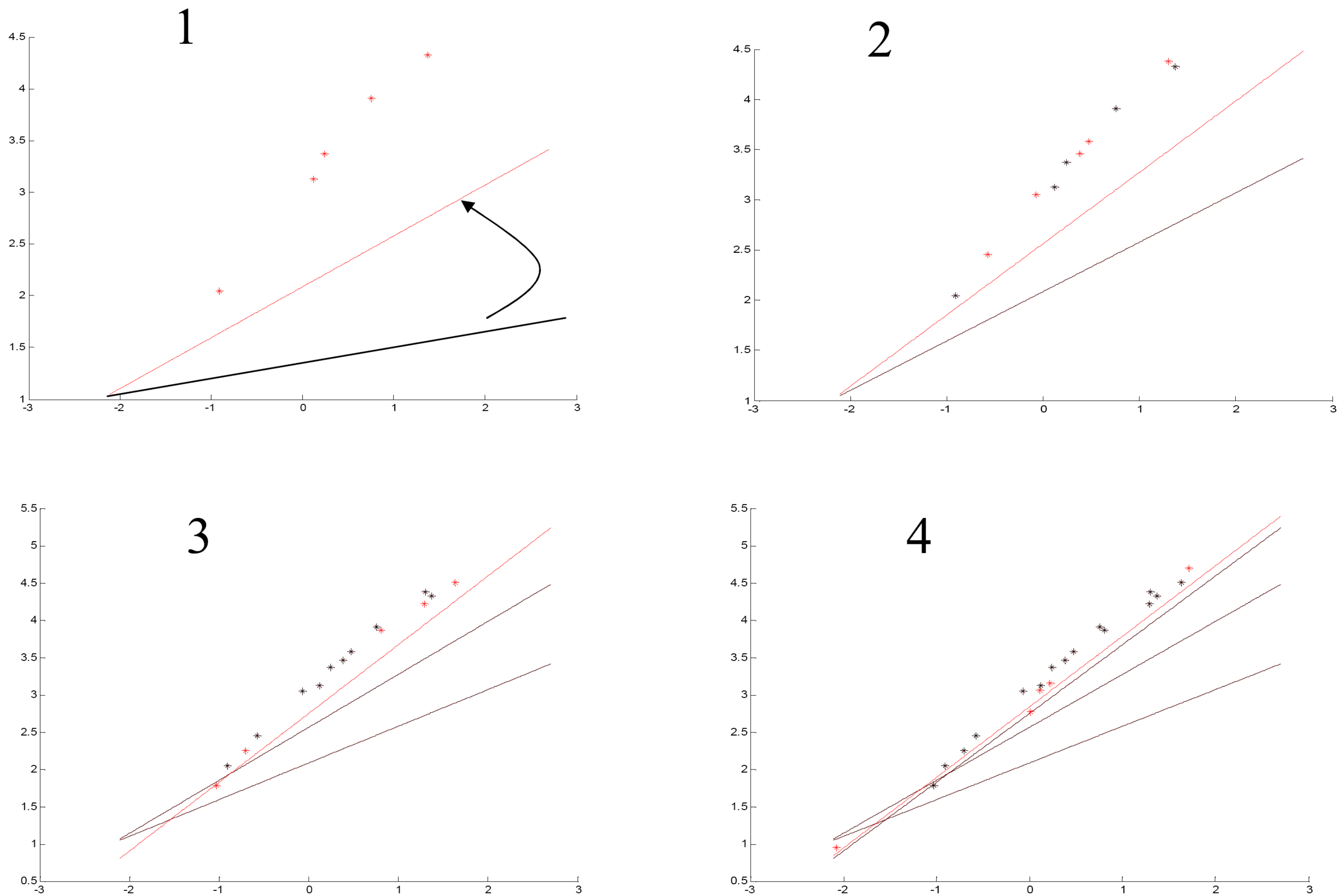
end for

return weights $\mathbf{w}$

- **Advantages:** very easy to implement, continuous data streams

Ref: Milos Hauskrecht

On-line Learning Example



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Linear Classification as a Linear Regression

2D Input space: $X = (X_1, X_2)$

Number of classes/categories K=3, So output $Y = (Y_1, Y_2, Y_3)$

Training sample, size N=5,

$$\mathbf{X} = \begin{bmatrix} 1 & x_{11} & x_{12} \\ 1 & x_{21} & x_{22} \\ 1 & x_{31} & x_{32} \\ 1 & x_{41} & x_{42} \\ 1 & x_{51} & x_{52} \end{bmatrix}, \quad \mathbf{Y} = \begin{bmatrix} y_{11} & y_{12} & y_{13} \\ y_{21} & y_{22} & y_{23} \\ \boxed{y_{31} \quad y_{32} \quad y_{33}} \\ y_{41} & y_{42} & y_{43} \\ y_{51} & y_{52} & y_{53} \end{bmatrix}$$

Each row has exactly one 1 indicating the category/class

Indicator Matrix

Regression output: $\hat{Y}((x_1, x_2)) = (1\ x_1\ \ x_2)(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{Y} = (x^T\beta_1\ \ x^T\beta_2\ \ x^T\beta_3)$

Bishop (3.35)

Or,

$$\hat{Y}_1((x_1\ x_2)) = (1\ x_1\ \ x_2)\beta_1$$
$$\hat{Y}_2((x_1\ x_2)) = (1\ x_1\ \ x_2)\beta_2$$
$$\hat{Y}_3((x_1\ x_2)) = (1\ x_1\ \ x_2)\beta_3$$

Classification rule:

$$\hat{G}((x_1\ \ x_2)) = \arg\max_k \hat{Y}_k((x_1\ \ x_2))$$

Diabetes Data Example

- **Diabetes data**

  The diabetes data set is taken from the UCI machine learning database repository at: http://www.ics.uci.edu/˜mlearn/Machine-Learning.html .

  The original source of the data is the National Institute of Diabetes and Digestive and Kidney Diseases. There are 768 cases in the data set, of which 268 show signs of diabetes according to World Health organization criteria. Each case contains 8 quantitative variables, including diastolic blood pressure, triceps skin fold thickness, a body mass index, etc.
  - Two classes: with or without signs of diabetes.
  - Denote the 8 original variables by $\tilde{X}_1, \tilde{X}_2, ..., \tilde{X}_8$
  - Remove the mean of $\tilde{X}_j$ and normalize it to unit variance.

Principle Features

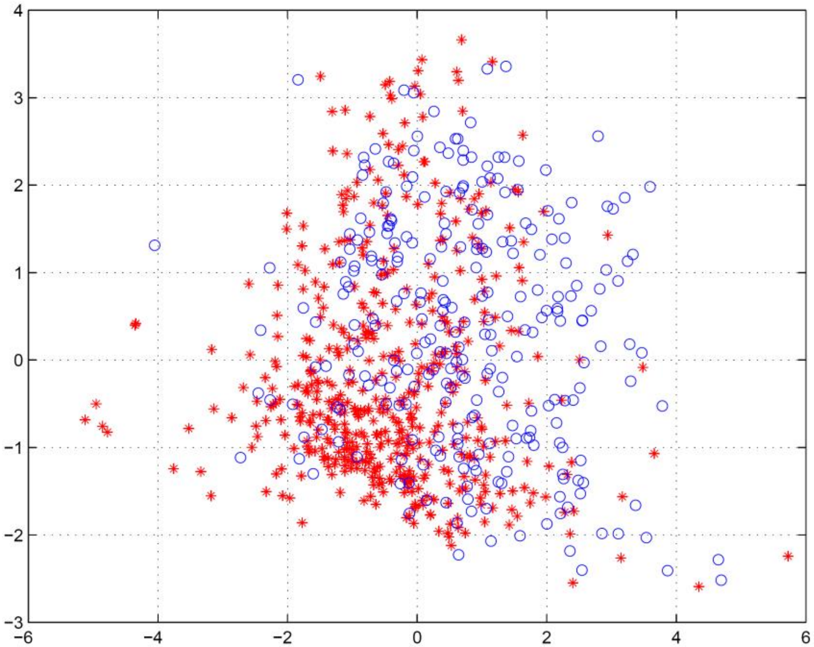
- The two principal components $X1$ and $X2$ are used in classification:

$$\begin{aligned} X_1 &= 0.1284\tilde{X}_1 + 0.3931\tilde{X}_2 + 0.3600\tilde{X}_3 + 0.4398\tilde{X}_4 \\ &\quad +0.4350\tilde{X}_5 + 0.4519\tilde{X}_6 + 0.2706\tilde{X}_7 + 0.1980\tilde{X}_8 \end{aligned}$$

$$\begin{aligned} X_2 &= 0.5938\tilde{X}_1 + 0.1740\tilde{X}_2 + 0.1839\tilde{X}_3 - 0.3320\tilde{X}_4 \\ &\quad -0.2508\tilde{X}_5 - 0.1010\tilde{X}_6 - 0.1221\tilde{X}_7 + 0.6206\tilde{X}_8 \end{aligned}$$

Linear Regression for Classification

- The scatter plot follows. Without diabetes: stars (class 1), with diabetes: circles (class 2).



$$\hat{\mathbf{B}} = (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{Y} = \begin{pmatrix} 0.6510 & 0.3490 \\ -0.1256 & 0.1256 \\ -0.0729 & 0.0729 \end{pmatrix}$$

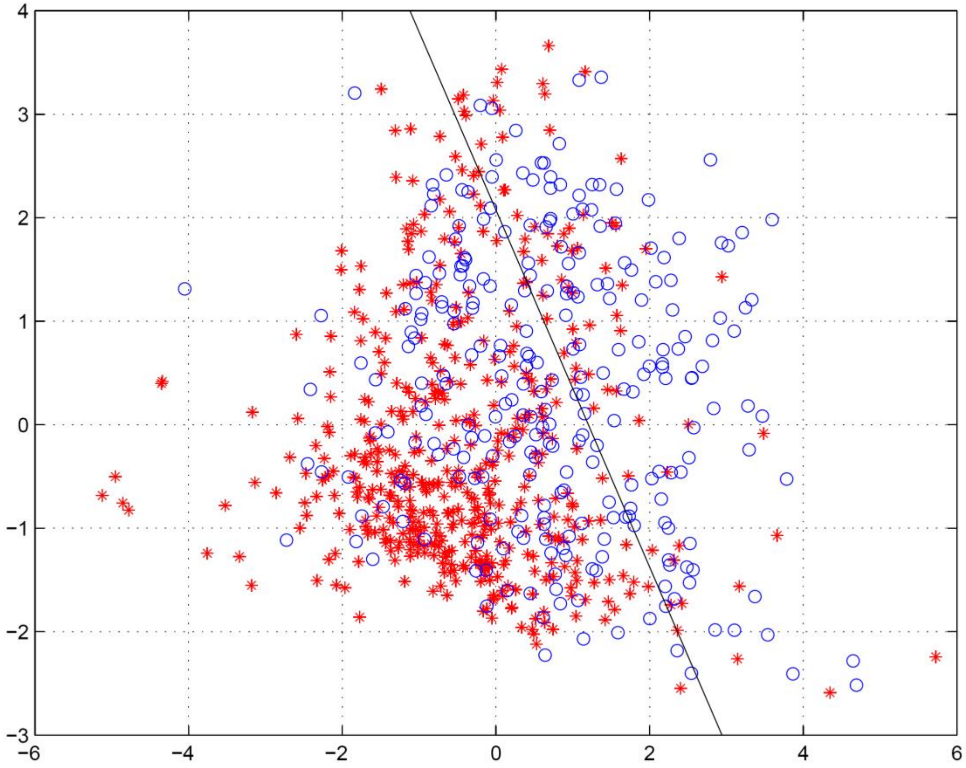
$$\hat{Y}_1 = 0.6510 - 0.1256X_1 - 0.0729X_2$$
$$\hat{Y}_2 = 0.3490 + 0.1256X_1 + 0.0729X_2$$

Classification Rule

$$\hat{G}(x) = \begin{cases} 1 & \hat{Y}_1 \geq \hat{Y}_2 \\ 2 & \hat{Y}_1 < \hat{Y}_2 \end{cases}$$

$$= \begin{cases} 1 & 0.151 - 0.1256X_1 - 0.0729X_2 \geq 0 \\ 2 & otherwise \end{cases}$$

- Classification error rate: 28.52%.



Linear Classification Discriminant Function

- There is a discriminant function $\delta_k(x)$ for each class k
- Classification rule: $R_k = \{x : k = \arg\max_j \delta_j(x)\}$
- In higher dimensional space the decision boundaries are piecewise hyperplanar