1. Which of the following are statements? (8 points)

   a) There is intelligent life elsewhere in the universe.
      Is a statement

   b) Your car is probably red.
      Is not a statement ("probably" does not have a truth value)

   c) $2 \times 5 = 25 - 17$
      Is a statement

   d) Is Klingon a real language?
      Is not a statement (questions cannot be statements)

2. Construct the truth tables for the following compound statements. Also indicate whether they are tautologies or contradictions (8 points)

   a) $A \rightarrow B \lor \overline{A}$

      | $A$ | $B$ | $\overline{A}$ | $B \lor \overline{A}$ | $A \rightarrow B \lor \overline{A}$ |
      |-----|-----|---------------|-----------------|-------------------------------|
      | T   | T   |   F           |     T            |    T                           |
      | T   | F   |   F           |     F            |    F                           |
      | F   | T   |   T           |     T            |    T                           |
      | F   | F   |   T           |     T            |    T                           |

      not a tautology or contradiction

   b) $A \land \overline{B} \lor A$

      | $A$ | $B$ | $\overline{B}$ | $A \land \overline{B}$ | $A \land \overline{B} \lor A$ |
      |-----|-----|---------------|-----------------|-------------------------------|
      | T   | T   |   F           |     F            |    T                           |
      | T   | F   |   T           |     F            |    T                           |
      | F   | T   |   F           |     T            |    T                           |
      | F   | F   |   T           |     T            |    T                           |

      is a tautology
3. Use propositional logic to prove that the following arguments are valid. (All steps in the proof sequence have to be annotated indicating the rule applied. Only the rules listed on the pages at the end of this exam can be used.) (16 points)

a) \( A \land (B \land A) \lor C \rightarrow B \lor C \)
1 \( A \) hyp
2 \( (B \land A) \lor C \) hyp
3 \( (B \land A) \land C \) 2 De Morgan
4 \( (B \land A) \) 3 sim
5 \( B \lor A \) 4 De Morgan
6 \( B \rightarrow \overline{A} \) 5 imp
7 \( \overline{B} \) 6, 1 mt
8 \( B \) 7 dn
9 \( B \lor C \) 8 add

b) \( (C \rightarrow B) \land \overline{A} \lor B \land ((A \land \overline{C}) \rightarrow D) \rightarrow A \land D \)
1 \( C \rightarrow B \) hyp
2 \( \overline{A} \lor B \) hyp
3 \( (A \land \overline{C}) \rightarrow D \) hyp
4 \( \overline{A} \land B \) 2 De Morgan
5 \( \overline{A} \) 4 sim
6 \( A \) 5 dn
7 \( \overline{B} \) 4 sim
8 \( \overline{C} \) 7, 1 mt
9 \( A \land \overline{C} \) 6, 8 con
10 \( D \) 9, 3 mp
11 \( A \land D \) 10, 6 con

4. Underline all instances of bound variables in the following expressions. (9 points)

a) \( (\forall x)(P(x) \lor (\exists y)(\forall z)R(y, z) \rightarrow Q(x, y)) \lor P(z) \)

b) \( (\exists z)(R(z) \land (\forall x)(P(x, z) \rightarrow Q(z) \lor R(z)) \lor Q(x)) \)

c) \( (\forall y)(Q(x, y) \lor (\exists x)R(x, y) \rightarrow P(x, y)) \lor (\exists z)P(y, z) \)

5. Use the predicates and constants

\( D(x) \) is ”\( x \) is a dog”
\( B(x) \) is ”\( x \) is a bone”
\( L(x, y) \) is ”\( x \) likes \( y \)”
\( E(x, y) \) is ”\( x \) eats \( y \)”
\( O(x, y) \) is "x owns y"
\( p \) is "Pluto"
\( j \) is "John"

to translate the following argument into predicate logic. (5 points)

John likes his dog Pluto and if Pluto owns a bone, Pluto eats it. All dogs like bones. Only if Pluto owns a bone does he like bones. Therefore Pluto eats a bone.

\[
\begin{align*}
D(p) & \land O(j, p) \land L(j, p) \land (\forall x)(B(x) \land O(p, x) \rightarrow E(p, x)) \land (\forall x)(\forall y)(D(x) \land B(y) \rightarrow L(x, y)) \land (\forall x)(B(x) \land L(p, x) \rightarrow (\exists y)(B(y) \land O(p, y))) \rightarrow (\exists x)(B(x) \land E(p, x))
\end{align*}
\]

6. Use predicate logic to prove the following argument. (All steps in the proof sequence have to be annotated by the rule used to derive it. Only the rules listed on the pages at the end of this exam can be used.) (10 points)

\[
(\exists x)(P(x) \lor R(x)) \land (\exists y)(Q(a, y) \rightarrow P(y)) \rightarrow (\exists x)(\exists z)(Q(x, z) \land \overline{R(x)})
\]

\[
\begin{align*}
1 & \quad (\exists x)(P(x) \lor R(x)) \quad \text{hyp} \\
2 & \quad (\exists y)(Q(a, y) \rightarrow P(y)) \quad \text{hyp} \\
3 & \quad \overline{Q(a, u)} \rightarrow P(u) \quad 2 \text{ ei} \\
4 & \quad (\forall x)P(x) \lor R(x) \quad 1 \text{ neg} \\
5 & \quad \overline{P(u)} \lor R(u) \quad 4 \text{ ui} \\
6 & \quad \overline{P(u)} \land R(u) \quad 5 \text{ De Morgan} \\
7 & \quad \overline{P(u)} \quad 6 \text{ sim} \\
8 & \quad Q(a, u) \quad 7, 3 \text{ mt} \\
9 & \quad Q(a, u) \quad 8 \text{ dn} \\
10 & \quad \overline{P(a)} \lor R(a) \quad 4 \text{ ui} \\
11 & \quad \overline{P(a)} \land R(a) \quad 10 \text{ De Morgan} \\
12 & \quad \overline{R(a)} \quad 11 \text{ sim} \\
13 & \quad Q(a, u) \land \overline{R(a)} \quad 12, 9 \text{ con} \\
14 & \quad (\exists z)(Q(a, z) \land \overline{R(a)}) \quad 13 \text{ eg} \\
15 & \quad (\exists x)(\exists z)(Q(x, z) \land \overline{R(x)}) \quad 14 \text{ eg}
\end{align*}
\]

7. For each of the following proofs indicate which proof technique was used (counterexample, exhaustive proof, direct proof, proof by contraposition, proof by contradiction, proof by induction). (12 points)

a) Conjecture: All positive integers, \( n \), that are not prime numbers are divisible by a number that is larger than \( \frac{5}{3}n \).

Proof: \( 25 \) is not a prime number and is only divisible by \( 5 \) which is not larger than \( \frac{1}{5}25 \). Therefore not all positive non-prime integers, \( n \), are divisible by a number larger than \( \frac{1}{5}n \).
Counterexample

b) Conjecture: If the sum of two different prime numbers is even then neither one of them is 2.
Proof: Assume the sum of two prime numbers, \( m \) and \( n \), is even (i.e. \( n + m = 2 \times k \)) and one of them is 2 (i.e. \( n = 2 \)). Then the other prime number would have to be \( m = 2 \times k - n = 2 \times k - 2 = 2 \times (k - 1) \) and thus even. However, since 2 is the only even prime number and the two prime numbers have to be distinct, neither of the two distinct prime numbers can be 2 if their sum is even.

Proof by contradiction

c) Conjecture: If the square of a number is even then the number is even.
Proof: Assume the number were odd. Then \( n^2 = (2 \times k - 1)^2 = 4 \times k^2 - 2 \times 2 \times k + 1 = 2 \times (2k^2 - 2 \times k) + 1 \) which is not even. Therefore if the square of a number is even, the number is even.

Proof by contraposition

d) Conjecture: All odd positive integers between 3 and 13 are either prime or square.
Proof: 3, 5, 7, 11, and 13 are prime numbers and \( 9 = 3^2 \) is a square number. Therefore all odd positive integers between 3 and 13 are either prime or square.

Exhaustive proof

8. Prove or disprove the following conjectures. You can use any of the proof techniques introduced in class. (16 points)

a) The sum of any positive integer and its square is even.

Case 1: the positive integer is even (i.e. \( m = 2 \times k \)).
Then the sum of it and its square is \( m + m^2 = m + m \times m = 2 \times k + 2 \times k \times 2 \times k = 2 \times (k + 2 \times k \times k) \) and thus even.

Case 2: the positive integer is odd (i.e. \( m = 2 \times k + 1 \)).
Then the sum of it and its square is \( m + m^2 = 2 \times k + 1 + (2 \times k + 1) \times (2 \times k + 1) = 2 \times k + 1 + 2 \times 2 \times k \times k + 2 \times 2 \times k + 1 = 2 \times (k + 2 \times k \times k + 2 \times k + 1) + 1 + 1 = 2 \times (k + 2 \times k \times k + 2 \times k + 1) \) which is even.

Therefore the sum of a positive integer and its square is even.

b) The product of two integers, \( n \) and \( m \), is less than or equal to one quarter times the square of their sum.

Base case: The difference between the two integers is \( k = 0 \) (i.e. \( n = m \)).
Then: \( \frac{(n+m)^2}{4} = \frac{(2n)^2}{4} = \frac{4n^2}{4} = n^2 = n \times m \)
Ind. Hyp.: The conjecture holds for two integers that have a difference of \( k \geq 0 \) (i.e. \( n = m + k \)).

Ind. Step.: The two integers have a difference of \( k + 1 \) (i.e. \( n = m + k \)).

Then:

\[
\begin{align*}
\frac{(n+m)^2}{4} &= \frac{(n+(m-1)+1)^2}{4} = \frac{(n+(m-1))^2+2(n+(m-1))+1}{4} = \frac{(n+(m-1))^2+2(n+(m-1))+1}{4} \\
n(m - 1) + \frac{2(n+(n-k-1)+1)}{4} &= n(m - 1) + \frac{4n-2k-2}{4} \geq n(m - 1) + \frac{4n}{4} = n \cdot m
\end{align*}
\]

9. Provide a recursive definition for the following sets. ( 8 points )

a) \( S = \{5, 10, 15, 20, 25, \ldots \} \)

\[
\begin{align*}
S(1) &= 5 \\
S(n) &= S(n - 1) + 5
\end{align*}
\]

b) \( S = \{ab, abab, ababab, abababab, ababababab, \ldots \} \)

\[
\begin{align*}
S(1) &= ab \\
S(n) &= S(n - 1)ab
\end{align*}
\]

10. Find (and prove) a closed form formula for the following recurrence relation using the expand, guess, and verify procedure. ( 8 points )

\[
P(1) = 1, P(n + 1) = P(n) + 5
\]

\[
P(n) = 5n - 4
\]

Base case: \( n = 1 \)

\[
P(1) = 1 = 1 \cdot 5 - 4
\]

Ind. Hyp.: \( P(n) = 5n - 4 \)

Ind. Step: \( P(n + 1) \)

\[
P(n + 1) = P(n) + 5 = (5n - 4) + 5 = 5(n + 1) - 4
\]