1. Rewrite the following sets as a list of elements. (9 points)
   a) \( \{ x | x \in \mathbb{N} \land x \text{ is a multiple of } 3 \land 1 \leq x \leq 12 \} \)
      \{3, 6, 9, 12\}
   b) \( \{ x | y \in \mathbb{N} \land x = y + y^2 \} \)
      \{0, 2, 6, 12, 20, 30, 42, ...\}
   c) \( \{ x | x \text{ is an even length string over the alphabet } \{a,b\} \land |x| < 5 \} \)
      \{aa, ab, ba, bb, aaaa, aaab, aaba, aabb, abaa, abab, abba, abbb, baaa, baab, baba, babb, bbba, bbab, bbba, bbbb\}

Rewrite the following sets by giving a characteristic property. (6 points)
   d) \( \{ Saturday, Sunday \} \)
      \( \{ x | x \text{ is a day of the weekend} \} \)
   e) \( \{ 5, 10, 15, 20, 25, ... \} \)
      \( \{ x | x \in \mathbb{N} \land x \text{ is a multiple of } 5 \land x \geq 1 \} \)

2. Determine which of the following are operations on the given set \( S \). You have to justify your decision (a short justification is sufficient). (6 points)
   a) \( x \circ y = x^2 + 5y \), \( S = \{ x | x \in \mathbb{Z} \} \)
      Is an operation since it is well defined (applies to all elements in \( \mathbb{Z} \)) and is closed (i.e. always results in an element in \( \mathbb{Z} \)).
   b) \( x \circ y = x \times y^2 - x^2 \times y \), \( S = \{ x | x \in \mathbb{N} \land x < 10 \} \)
      Is not an operation since it is not closed on the natural numbers smaller than 10 (the result can be a negative number - or be larger than 10)
   c) \( x^# = axb \), \( S = \{ x | x \text{ is a palindrome over the english alphabet} \} \)
      Is not an operation since it is not closed on the set of palindromes (the resulting string is not a palindrome since it starts with \( a \) and ends in \( b \).
3. Prove the following subset relation (Hint: To prove equality $A = B$ without equality rules we actually proved two subset relations $A \subseteq B$ and $B \subseteq A$. To prove a subset relation you only have to do one of the two directions.). (4 points)

$$\overline{A \cup B} \cup B \subseteq \overline{A \cup B}$$

$$x \in (A \cup B \cup B)$$
$$\rightarrow x \in \overline{A \cup B} \vee x \in B$$
$$\rightarrow x \notin (A \cup B) \vee x \in B$$
$$\rightarrow (x \notin A \wedge x \notin B) \vee x \in B$$
$$\rightarrow (x \notin A \vee x \in B) \wedge (x \notin B \vee x \in B)$$
$$\rightarrow (x \notin A \vee x \in B) \wedge T$$
$$\rightarrow x \notin A \vee x \in B$$
$$\rightarrow x \in \overline{A} \vee x \in B$$
$$\rightarrow x \in \overline{A \cup B}$$

Prove the following set identity using the basic set identities listed in the table at the end of the exam. At every step indicate which identity rule was used. (5 points)

$$((A \cup B) \cap (B \cup \overline{A})) \cap (\overline{B} \cup A)) = A \cap B$$

$$((A \cup B) \cap (B \cup \overline{A})) \cap (\overline{B} \cup A)$$
$$ass$$

$$((A \cup B) \cap (\overline{A} \cup B)) \cap (\overline{B} \cup A)$$
$$com$$

$$((A \cap \overline{A}) \cup B) \cap (\overline{B} \cup A)$$
$$dist$$

$$(\emptyset \cup B) \cap (\overline{B} \cup A)$$
$$comp$$

$$B \cap (\overline{B} \cup A)$$
$$id$$

$$(B \cap \overline{B}) \cup (B \cap A)$$
$$dist$$

$$\emptyset \cup (B \cap A)$$
$$comp$$

$$B \cap A$$
$$id$$

4. Show that the following sets are denumerable. (8 points)

a) $\{x| x \in \{Tuesday, Thursday\} \times \mathbb{Z}\}$

This set can be enumerated as follows: $\{(Tuesday, 0), (Thursday, 0), (Tuesday, -1), (Thursday, -1), (Tuesday, 1), (Thursday, 1), (Tuesday, -2), (Thursday, -2), ...\}$

This corresponds to the enumeration function (assuming that $i$ starts from 0):

$$S(i) = \begin{cases} 
(Tuesday, (-1)^{\frac{i}{2}} \lceil \frac{i+2}{4} \rceil) & \text{if } i \text{ is 0 or even} \\
(Thursday, (-1)^{\frac{i}{2}} \lceil \frac{i+2}{4} \rceil) & \text{if } i \text{ is odd}
\end{cases}$$
b) \(\{(x, y) \mid x \in \mathbb{Z} \land \frac{x}{2} \in \mathbb{Z} \land y \in \mathbb{N}\}\)

This set can be enumerated as follows: \((0, 0), (-2, 0), (0, 1), (0, 2), (-2, 1), (2, 0),\)
\((-4, 0), (2, 1), (-2, 2), (0, 3), (0, 4), (-2, 3), (2, 2), (-4, 1), \ldots\)

This can be illustrated using dovetailing as follows:

\[
\begin{array}{ccccccc}
(0, 0) & (0, 1) & (0, 2) & (0, 3) & (0, 4) & \ldots \\
(-2, 0) & (-2, 1) & (-2, 2) & (-2, 3) & (-2, 4) & \ldots \\
(2, 0) & (2, 1) & (2, 2) & (2, 3) & (2, -2) & \ldots \\
(-4, 0) & (-4, 1) & (-4, 2) & (-4, 3) & (-4, 4) & \ldots \\
(4, 0) & (4, 1) & (4, 2) & (4, 3) & (4, 4) & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}
\]

5. Use the addition principle, multiplication principle, and the principle of inclusion and exclusion to calculate the solution to the following problems (You only have to give the formula. The final number is not necessary.). (8 points)

a) Given the sets \(A = \{\text{red, green, blue}\}, B = \{\text{blue, purple, black}\},\) and \(C = \{\text{blue, red, yellow}\},\) how many pairs of the available colors are there that contain at least one primary color (i.e. how many elements are there in the set \((A \cup B \cup C) \times A)\)?

\[|A \cup B \cup C| \times |A| = (|A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|) \times |A| = (3 + 3 + 3 - 1 - 2 - 1 + 1) \times 3 = 6 \times 3 = 18\]

b) A car dealer sells 5 models of trucks and 7 models of cars and offers each vehicle in 6 different colors. How many different vehicle configurations can be ordered from the dealer?

\[(5 + 7) \times 6 = 12 \times 6 = 72\]

Indicate for the following problem if it is a permutation or combination problem and determine the number of possibilities. (The formula is sufficient. The final number does not have to be calculated.) (5 points)

c) How many different ways are there to distribute 20 pennies among 5 children?

This is a combination problem with duplicates since the order does not matter and it does not matter which pennies are given to which child. Its solution takes the following form:

\[
\frac{(20+4)!}{20!4!} = 10,626
\]

6. Determine for the following relations over the set \(S = \{1, 2, 3, 4, 5, 6\}\) if they are reflexive, symmetric, antisymmetric, or transitive. (6 points)

a) \(\rho = \{(2, 2), (4, 4), (6, 6)\}\)

symmetric, antisymmetric, transitive
b) \( x \rho y \leftrightarrow x + y \) is even  
\[ \text{symmetric, reflexive, transitive} \]

7. Form the respective closures for the following relations over the set \( S = \{1, 2, 3, 4, 5, 6\} \). (8 points)

a) Form the symmetric and transitive closure of \( \rho = \{ (1, 2), (2, 2), (3, 2), (5, 3), (6, 6) \} \).
\[ \rho^* = \rho \cup \{ (2, 1), (2, 3), (3, 5), (1, 1), (3, 3), (5, 5), (2, 5), (5, 2), (1, 3), (3, 1), (1, 5), (5, 1) \} \]

b) Form the closures necessary to transform the relation \( \rho = \{ (3, 1), (2, 2), (4, 6), (5, 3), (6, 2) \} \) into an equivalence relation.
\[ \rho^* = \rho \cup \{ (1, 1), (3, 3), (4, 4), (5, 5), (6, 6), (1, 3), (6, 4), (3, 5), (2, 6), (1, 5), (5, 1), (2, 4), (4, 2) \} \]

8. For each of the following functions indicate if they are one-to-one, onto, or bijective. (9 points)

a) \( f : \mathbb{Z} \rightarrow \mathbb{R} \), \( f(x) = x + 3 \)
\( \text{Is one-to-one} \)

b) \( f : \mathbb{Z} - \{0\} \rightarrow \mathbb{N} - \{0\} \times \{-1, 1\} \), \( f(x) = (|x|, \text{sign}(x)) \)
\( \text{Is bijective} \)

b) \( f : \{\text{white, red, green, blue}\} \rightarrow \{\text{pure, mixed}\} \), \( f = \{ \text{(white, mixed), (red, pure), (green, pure), (blue, pure)} \} \)
\( \text{Is onto} \)

9. Rewrite the following permutations on \( S = \{1, 2, 3, 4, 5, 6, 7\} \) in cycle notation. (8 points)

a) \( f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 2 & 5 & 6 & 1 & 4 & 7 \end{pmatrix} \)
\( (1, 3, 5) \circ (4, 6) \)

b) \( f = \{ (1, 2), (2, 7), (3, 6), (4, 1), (5, 5), (6, 3), (7, 4) \} \)
\( (1, 2, 7, 4) \circ (3, 6) \)

10. Prove the following orders of magnitude by specifying adequate constants. (8 points)

a) \( f(x) = 2 \times x^4 + 3 \times x^2 - 3x \), \( f = \Theta(x^4) \)
\( c_1 = 2, c_2 = 3, n = 3, c_1 x^4 \leq 2 \times x^4 + 3 \times x^2 - 3x \leq c_2 x^4 \text{ for } n > 3 \)

b) \( f(x) = 3 \times 2^x - 10x \), \( f = \Omega(2^x) \)
\( c = 2, n = 6, c2^x \leq 3 \times 2^x - 10x \text{ for } n > 6 \)

11. Give the results of the following matrix operation if it is possible (if the operation is not possible, justify why). (5 points)
a) \[
\begin{pmatrix}
1 & 0 & -1 \\
0 & 1 & 1
\end{pmatrix}
\times
\begin{pmatrix}
1 & 2 \\
0 & 0 \\
-1 & 1
\end{pmatrix}
+ \begin{pmatrix}
1 & -1 \\
1 & 0
\end{pmatrix}
= \begin{pmatrix}
2 & 1 \\
-1 & 1
\end{pmatrix}
+ \begin{pmatrix}
1 & -1 \\
1 & 0
\end{pmatrix}
= \begin{pmatrix}
3 & 0 \\
0 & 1
\end{pmatrix}
\]

Give the results of the following boolean matrix operations. (5 points)

b) \[
\begin{pmatrix}
1 & 1 & 0 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{pmatrix}
\lor
\begin{pmatrix}
1 & 1 & 0 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{pmatrix}
\times
\begin{pmatrix}
1 & 1 & 0 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{pmatrix}
= \begin{pmatrix}
1 & 1 & 1 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{pmatrix}
\lor
\begin{pmatrix}
1 & 1 & 1 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{pmatrix}
= \begin{pmatrix}
1 & 1 & 1 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{pmatrix}
\]
Basic Set Identities

For the set identity proof you can use only the rules given in the following table. All other rules have to be proven first.

<table>
<thead>
<tr>
<th>Identity</th>
<th>Property</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A \cup B = B \cup A$</td>
<td>Commutativity</td>
</tr>
<tr>
<td>$A \cap B = B \cap A$</td>
<td></td>
</tr>
<tr>
<td>$(A \cup B) \cup C = A \cup (B \cup C)$</td>
<td>Associativity</td>
</tr>
<tr>
<td>$(A \cap B) \cap C = A \cap (B \cap C)$</td>
<td></td>
</tr>
<tr>
<td>$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$</td>
<td>Distributivity</td>
</tr>
<tr>
<td>$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$</td>
<td></td>
</tr>
<tr>
<td>$A \cup \emptyset = A$</td>
<td>Identity</td>
</tr>
<tr>
<td>$A \cap S = A$</td>
<td></td>
</tr>
<tr>
<td>$A \cup \overline{A} = S$</td>
<td>Complement</td>
</tr>
<tr>
<td>$A \cap \overline{A} = \emptyset$</td>
<td></td>
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</tbody>
</table>