Combinatorial Pawn Power

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Abstract

This paper extends some of the basic results on chess endgames. In particular we analyze five famous endgames in the history of chess. We propose some new analytical constructs that help in understanding the outcome of the endgames and accurately project the winners. The generalized analysis of endgames enable us to obtain some remarkable game position values such as: -1, $-\frac{1}{2}$, $\frac{3}{2}\downarrow$, $\downarrow\star$, $\downarrow$, 0, $\star$, $\star 2$, $\uparrow^{-2}$, $\uparrow$, $\{0,\uparrow\star \mid 0\}$, $1\uparrow$, 1, 2, 3, *dud* , *over* and *under*.

1 The Buildup

The *endgame* is the final phase of a chess game. Most spectators find these final few moves as boring and unattractive. However, it is these moves that often decide the winner of a game.

The endgame is vastly neglected because much emphasis is put on identifying positive and strong opening manoeuver(s). Chess is such a complex game that it would be useless not to study all the aspects of the game. Endgames in particular are of interest to the combinatorial game theoreticians. The reasons are simple: 1) endgames offer a short range analysis on the outcome of chess games, 2) endgames can easily be expressed using combinatorial game theory, and 3) they offer new and interesting positions for the advancement of the infant theory.

A natural question at this point would be: when an endgame begins? The answer to this is not known — it is sometimes a matter of opinion. There is no clear division between the middle game and the endgame. To better understand this we quote from literature the following passage [7, pp. 10–11]:

As a rule of thumb, the endgame begins when the queens are gone. Take the Berlin Defense to the Ruy Lopez that was rehabilitated by Vladimir Kramnik against Gary Kasparov in their 2000 title match: 1. `e4 e5` 2. `Nf3 Nc6` 3. `Bb5 Nf6` 4. `0-0 Nxe4` 5. `d4 Nd6` 6. `Bxc6 dx6` 7. `dxe5 Nf5` 8. `Qxd8+ Kxd8`.

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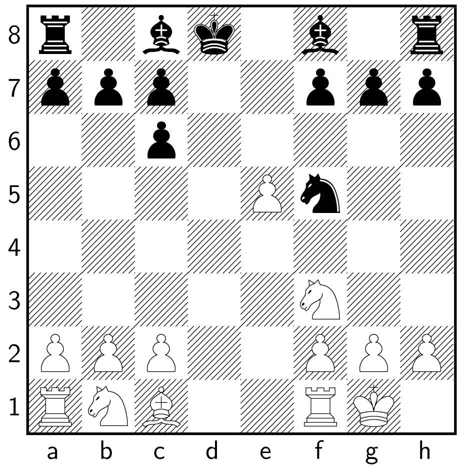

Figure 1: Position after 8 $\cdots$ `Kxd8`.

You could call this (Figure 1) a middle game or an endgame even though most of the men (pieces) are still on the board. Is it the beginning of the end or the end of the beginning?

Previous work on chess endgames can be found in [5] and [6], which inspired numerous endgame analysis including [4], [8], [9], etc. Our work differs from all the above in a number of ways. For example: 1) analysis in [5] and [6] assume some what stagnant Kings and rely on mutual zugzwang plays, 2) work reported in [8] and [9] analyze Xiangqi with numerous superfluous assumptions, while 3) [4] is a specific $\star 2$ position analyzer with no real endgame analysis. We undertake no assumptions on the piece movements are undertake analysis of real (recorded) chess endgames.

This paper aims to provide sound and concrete analysis of chess endgames in a discreet stepwise fashion. The ultimate goal is to mature the analytical tools on similar lines as that of Go [2]. We encourage the readers to see [8] for a basic introduction to combinatorial game theory, and [5] for its elegant application to chess endgames.

The rest of the paper is organized as following. Section 2 introduces some basic tools necessary to explore the game positions in complex chess endgames. In Section 3 we focus on projecting the outcomes of five famous chess endgames recorded in history, followed by some concluding remarks in Section 4.

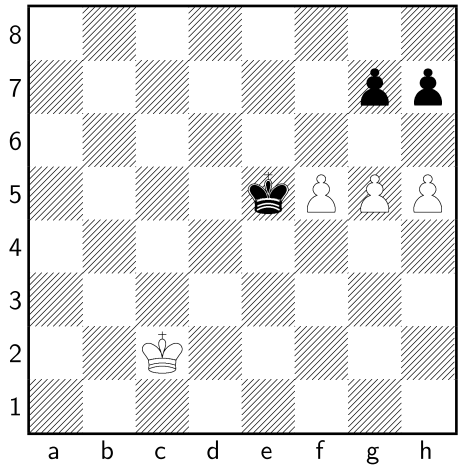

Figure 2: White to play `f6` and win.

2 The Lineup

Before we analyze championship endgames, it would be ideal to explore a simple game and see the strength that a simple pawn piece possesses. This section will also encapsulate the few drawbacks that were identified in the previous work reported on chess (see Section 1).

Figure 2 shows a game where White has the initial move. It is obvious that Kings do not play any role in the immediate outcome of the game since they are far apart from each other. The three possible moves for White are: 1. `h6`, 1. `g6`, and 1. `f6`. We should note here that our analysis are in a discreet stepwise mode, whereas analysis undertaken in [5], [6], [8] and [9] were targeted towards a generalized outcome of the game. We shall soon see that our proposed method is simpler and has the same effectiveness as the previous reported work.

File h:

1. `h6? gxh6`. The subgame has a value of 0, i.e., $\{\,|\,\}$. To see why this is a second player win, it is enough to observe the symmetrical pawn positions on files *g* and *h*. If Black was to move first, it would also result in a game value of $\{\,|\,\}$, i.e., 1. `h6? gxh6`.

2. `g6 hxg6`. White chooses to sacrifice its pawn in order to remove Black's threat to its pawn on file *h*. It is to be noted that this subgame is a win for Black, even when White delays to play `g6`. The subgame has a value of $\{\star\,|\,0{,}0\}$. White's move to `g6` transforms the subgame to first player win, i.e., $\star$. Black on the other hand has two moves (`Kxf5` and `hxg5`). Both have an outcome of 0 because the pieces are captured and the subgame vanishes immediately (a very common

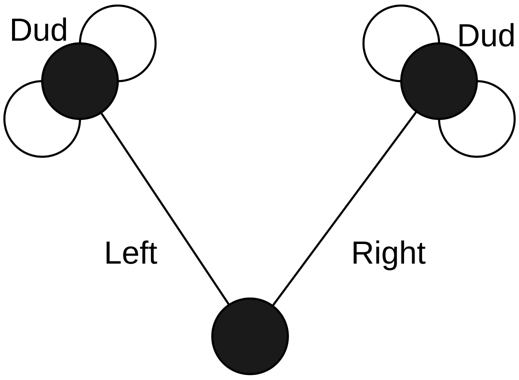

Figure 3: Game tree representing a draw in chess.

assumption in endgame analysis for details see [5]). Together the subgame has a value of $\{\star \mid 0{,}0\} = \{\star \mid 0\} = -\uparrow = \downarrow$. $\downarrow$ is an infinitesimally small negative value less than 0, and it means there is an apparent advantage for Black at this point. These infinitesimal advantages aggregate over a period of time to mature into a clear benefit.

3. `fxg6 Kf6`. White captures pawn on file *g* to reduce the Black's pawn advantage (subgame value 0). Black on the other hand has two move both of value 0, i.e., it can either capture on `Kxf5` or gxf5. The subgame has a value of $\{0 \mid 0{,}0\} = \{0 \mid 0\} = \star$ — the first player to move wins.

4. `Kd3`. Black moves diagonally up to strategically force White to abandon the game. At this point we introduce a special value for the subgame that ends in a draw or seems to end in a draw. These types of games are not discussed in [1] (the "Bible" of combinatorial game theory), for the strict reason that a definitive winner should emerge in a combinatorial game [1, p. 17]. Over the number of years this assumption is somewhat relaxed to include ties [9]. However, a majority of chess games do indeed end in a draw. We recall from [1] the subgame value of *dud*, where no player is the apparent winner but play can go on indefinitely. The game tree of *dud* is represented in Figure 3. The game tree analysis show that the players in order to avoid loss, switch back and forth between: 1) the position which starts the *dud* game value and 2) a temporary position that is used to explore all the (vain) options.

We now sum up all the subgame values to come up with a conclusive outcome of the game. White to move: 1. `h6? gxh6` 2. `g6 hxg6` 3. `fxg6 Kf6` 4. `Kd3` is equivalent in combinatorial game theory representation: 1. $\{ \mid \}$ 2. $\{\star \mid 0\}$ 3. $\{0 \mid 0\}$ 4. Dud. Therefore, the aggregate value of the endgame represented in Figure 2 is: $0 + \star + \downarrow + \mathit{dud} = \mathit{dud}$ — a draw.

File g:

1. `g6? h6`. Again the symmetry of the board reduces the subgame to $\{\star \mid \star\} = 0$. Both White and Black have only one move on file *g*, which forces the opponent to choose a symmetrical move on file *h*.

2. `Kd3 Kxf5`. After the initial move, White has no option than to mobilize its King. This results in a loss of one move (in favor of Black). This move could have been vital in White's survival. If at this stage White would have been able to move its *h*–pawn, it could have raced the King to the pawn's queen–ship. The subgame has a value of $\{-1 \mid 0\} = -\frac{1}{2}$. Therefore, the entire game value is: $0 + -\frac{1}{2} = -\frac{1}{2}$ — a half move advantage for Black to secure a win.

File f:

1. `f6? gxf6`. White moves its pawn for sacrifice and creates a game value of 0. If Black had an opening move, it would also had to sacrifice one of its pawns. Thus, the subgame has a value of $\star$.

2. `g6! hxg6`. To create a pass for its pawn on file *h*, White sacrifices another pawn on file *g*. A similar value of subgame is achieved as in Step 1.

3. `h6!`. White now has a passable pawn on file *h*. Ultimately in successive moves $\cdots$ `Qh5!` White will seal the fate of its opponent. Interesting to see is that after playing 3. `h6!`, the game value is only $\{1 \mid \} = 2$ — an advantage of two spare moves for White. The entire game's combinatorial value is: $0 + 1 + 2 = 3$.

We have effectively applied a discrete analytical combinatorial game theoretical analysis to a chess endgame. At first it seems tedious and lengthy analysis, but endgames are worth that attention. As we have seen if a wrong move is made (`g6`), it could cost White the game.

3 The Wins

We now proceed with some of the finest chess endgames recorded in history. These games range form a postal game to world championships. All of these games have a different and exciting outcome. In all cases we show the actual game moves. Surprisingly, using our analytical methods, we identify shorter games and correct some of the fatal mistakes.

3.1 Mason vs. Englisch

Figure 4 shows the endgame between Mason and Englisch played in London 1883. The strategic location of pawn at `h2` is exploited.

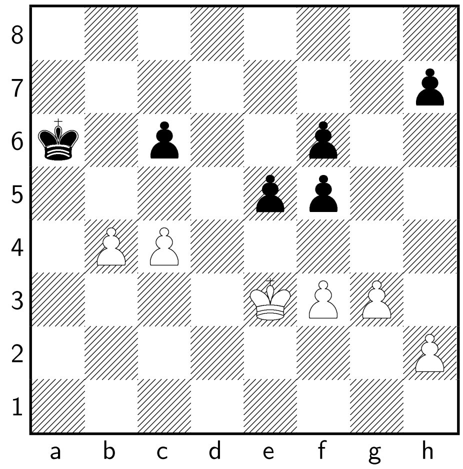

Figure 4: Mason vs. Englisch: White to play h3 and win.

1. h3!h5. Both White and Black have two separate moves to play. To understand the game value of file *h*, we analyze only the moves of White (moves made by Black pawn are symmetrical and will result in the exact same subgame values). If White plays h3, Black has in turn two options h6 would result in 0, and h7 would result in a $\star$. Thus, the overall value for file *h* is: $\{0,\star \mid 0,\star\} = \star 2$. Authors in [4] and [6] made a huge effort in constructing this position, which could have been obtain by analyzing the above position. Readers who are interested in the significance of the value $\star 2$ are encouraged to see [3, pp. 122–127]. In simple words star–two ($\star 2$) can be interpreted as a delaying mechanism — wait and see moves. Unfortunately this work both ways so at this moment no player has a significant advantage. Note that on file *h* Black (after the first move (h5)) has less delay than White.

2. g4 fxg4. Now, the real game begins. White moves its pawn to g4 and offers a sacrifice. The subgame effecting these two pawns is a $\star$.

3. fxg4 hxg4. The sacrifice pays off White captures and threatens Black pawn on h5. The subgame value is a $\star$.

4. h4!. White wins since this passed pawn cannot be stopped. The value of the game in Figure 4 is $\star 2 + \star + \star = \star 2$. The difference between White and Black is how well do they play their pawns on file *h*.

A win is also obtained via a longer sequence of moves: 1. $\cdots$ Kb6 2. g4 fxg4 3. hxg4 $\cdots$ Ke4-f5.

In reality the game continued 1. g4? fxg4! (but Black lost by 1. $\cdots$ f4+ 2. Ke4 h6 3. h4 Kb6 4. g5 fxg5 5. hxg5 hxg5 6. Kxe5 g4 7. Kxf4 gxf3 8. Kxf3 Kc7 9. Ke4 Kd6 10. Kf5) 2. fxg4 h6 3. h4 Kb6 4. Ke4 Kc7 5. g5 fxg5 6. hxg5 hxg5 7. Kxe5 g4 8. Kf4 Kd6 9. Kxg4 Ke5=.

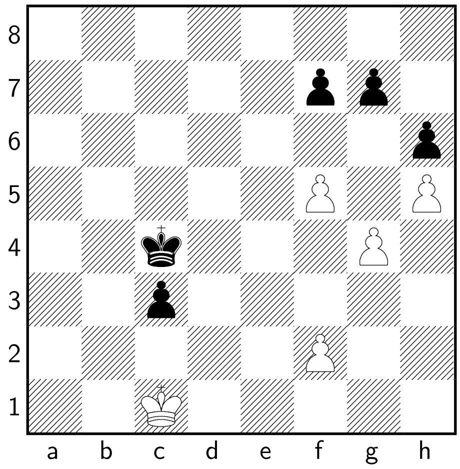

Figure 5: Ed. Lasker vs. Molle: White to play `f6` and win.

3.2 Ed. Lasker vs. Molle

Figure 5 shows the endgame between Ed. Lasker and Molle played in Berlin 1904. The strategic location of pawn at `f5` is exploited.

1. `f6! gxf6`. White forces Black to capture its pawn on `f6` to avoid a delayed queen–ship of the same pawn on file *g*. This fatal move will ultimately stop its two pawns on file *f*. The game value on file *f* alone is $\{0,\star \mid \star \parallel 0 \mid 0\} = \uparrow^{-2}$. This subgame value is smaller than $\uparrow$, and exhibits a very small advantage for White [3]. This is true since White's preferable move is to sacrifice its pawn on `f6`, it should not be the objective of Black (under any circumstances). We will now describe how we obtain the value of $\uparrow^{-2}$.

White moves to `f6`. The subgame value at this point is a $\star$, i.e., `gxf6` and `fxg7`. White can also move its bottom pawn to `f3`. In that case Black responds with `f6`. White now forces Black to sacrifice its pawn on file *g*. On the other hand if White uses its en–passant move, it axes its own foot by positioning itself for a wrong sacrifice on `g5`. Therefore, with White to move first, the subgame has a value of $\{0,\star \parallel 0 \mid 0\} = \uparrow$. Black's move of `g6` is the only one that is pending analysis. That move would result in a value $\star$. The analysis of a symmetrical position has already been provided (when White moves its upper pawn to `f6`). Thus, the total value of the subgame becomes $\{0,\star \mid \star \parallel 0 \mid 0\} = \uparrow^{-2}$.

2. `f4 Kd4`. After sacrificing its pawn on `f6`, White has an infinitesimally small advantage of $\uparrow^{-2}$, which it consolidates by moving its pawn to `f4`. White's move leaves Black with no option than to mobilize its King. If it moves any of its pawns, White will have a premature triumph on file *h*. The rational behind Black's move is simply to wait and see if White makes a mistake. The combinatorial value of this subgame is $\{0 \mid \} = 1$ — White buys an extra move.

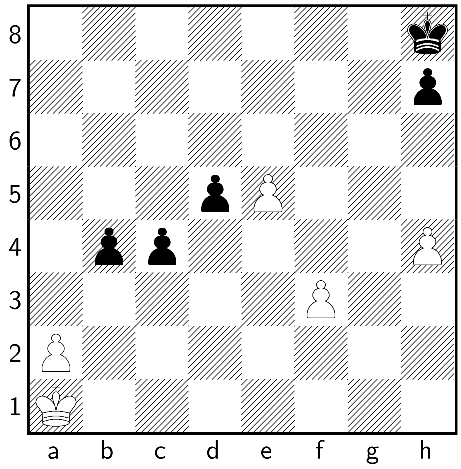

Figure 6: Taimanov vs. Botvinnik: Black to play d4 and win.

3. g5! fxg5. The move of g5 transforms the game into a value of $\star$. For Black it really does not matter which pawn it uses for capture (symmetry).

4. fxg5 Ke5. White's best move is to capture the Black pawn on g5. In doing so it provokes Black to capture in retaliation. But Black's best strategy is to keep the King moving diagonally up, so that it can squeeze the pawns on file *h*. The subgame has a value of $\{\star \mid \} = 0$, i.e., a second player win.

5. gxh6 Kf6. White's capture leaves Black the golden opportunity to tighten the blockade on file *h*, which it immediately does. The subgame at this point has a value of 0.

6. Kc2!. Step 5 emphasizes that White can be triumphant if it can force Black to make a first move towards file *h*. Black of course would not want that to happen, and would retreat. To achieve this, White mobilizes its King. Black is left with no option but retreat. The *h*–pawn will queen after Black's King abandons f6. The game value combined from all the combinatorial analysis is: $\uparrow^{-2} + \star + 0 + 0 = \{0,\uparrow \star \mid 0\}$ — White has a strategically infinitesimally small advantage over Black. However, Black's only chance to win in this game seems to wait for its opponents mistake, i.e., it waits to cash on second player win apparent form its game value of 0, which never happens.

In the actual game White resigned after 1. f4? f6 2. g5 Kd4.

3.3 Taimanov vs. Botvinnik

Figure 6 shows the endgame between Taimanov and Botvinnik played in Moscow 1953. The pawn at position d5 is exploited.

1. $\cdots$ d4!. The pawn at d5 is the only viable move for Black. This is due to the

fact that pawns on files *b* and *c* are too close to the White's King. Moving `d5` will distract White to make a rush for queen–ship on file *e*. On the other hand White's best shot is to make Black's King remain stagnant. We should also note that combinatorial game theory is not sufficient enough to deal with such an open board. Some will argue that in [5] open boards were analyzed. However, those boards were strategically chosen so that pawns could be stopped. That indeed helps in analysis but it negates the purpose of a concrete application of combinatorial game theory to chess. This drawback was realized in [8] and [9]. Nevertheless, we will pose this initial board setup as an exercise for the readers (see Section 4).

2. `e6 Kg7`. As predicted after the first move, White makes a run for queen–ship. Black moves its King for blockade. This position was previously analyzed in Section 3.1, and has a value of $\{\star \mid \star\} = 0$.

3. `f4 Kf6`. White moves its pawn on `f4` in order to make a double penetration. However, this is countered by Black's move of `Kf6`. White has basically axed its own foot. An advantage of 0 (Step 2), is now transformed into a subgame position of value $\star$.

4. `f5 d3`. A move of `f3` gives White a strategic advantage to threaten the Black King, and queen one of its two pawns (on files *e* and *f*). Black on the other hand responds by drawing attention to its run to queen–ship by making a move to `d3`. Files *e* and *f* now have a subgame value of $\{0 \mid \star, 0\} = \downarrow\star$.

5. `Kb2 h5`. White now focuses on blocking Black pawns. Black counters that with an en–passant move to block White pawn on `h4`. File *h* has a value of 0.

6. `Kc1 b3!`. White positions its King to stop the pawns, while Black maneuvers to overwhelm Black by posting `b3`. The subgame value for files *b* and *c* at this point is $\{0 \mid 1\}$ — Right has a win no mater who moves first (a $\frac{1}{2}$ move advantage see Section 3.2 for more details).

7. `a4 c3`. White's only hope is to see if Black makes a mistake (a rare thing to ask for, when Black is so much under control) and plays `a4`. Black's response strengthen's its position on files *b* and *c* to $\{0 \mid 2\}$ — a one whole move advantage.

The complete combinatorial value of the game described in Figure 6 is $0 + \star + \downarrow\star + 0 + \{0 \mid 1\} + \{0 \mid 2\} = \frac{3}{2}\downarrow$.

In the actual game White resigned after 5. `Kb2 h5`.

3.4 Muller vs. Rhode

Figure 7 shows the endgame between Muller and Rhode, a postal game played in 1897. The pawn at position `b7` is exploited.

1. $\cdots$ `b6`. Black's main focus is to counter the threat of queen-ships by White pawns. A play of `b6` forces a mutual cleanup. The board position after this move was already analyzed in Section 3.1, and had a value of $\{\star \mid \star\} = 0$.

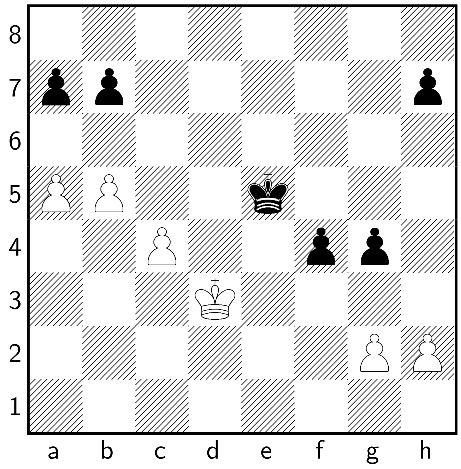

Figure 7: Muller vs. Rhode: Black to play `b6` and win.

2. `axb6 axb6`. Tit for tat captures are made which gives the subgame a value of $\{0 \mid 0\} = \star$.

3. `c5 bxc5`. Another sacrifice is offered to have the *b*–pawn queened. The subgame changes dramatically in favor of White with a game value of $\{2 \mid \}$, i.e., 3 spare moves.

4. `b6 Kd6` 5. `b7 Kc7` 6. `b8/Q+ Kxb8`. White's only hope is to use the three spare moves and louvre Black's King in the pursuit, and then the White's King can launch an offensive against the Black pawns. It is to be noted that such moves (4 through 6) have never been analyzed using combinatorial game theory. The reasons are unknown but analysis reveal a simple value of $\{0 \mid 0\} = \star$. The readers can easily verify this by translating the diagonal path to an imaginary file $b{+}\epsilon$, i.e., a file between files *b* and *c*.

7. `Ke4 Kc7`. White makes a run to capture *f*–pawn. Black counters this by blockading White pawn approach. In order to do this Black should reach `Kd3`. Black has three spare moves to do that ($\{ \mid 2\}$). This is true incase White decides not to move its King after the capture at file *f*. Subgame value for files *e* and *f* is $\{0 \mid \}$ (spare move for White). Therefore, the current subgame has a combined value of $\{0 \mid \| \mid 2\} = \{0 \mid 2\}$.

8. `Kxf4 h5`. White captures a pawn and immediately threatens *g*–pawn. This is countered by Black's en–passant move. The subgame value of files *f* and *g* is again $\{0 \mid \}$. On the other hand file *h* (considering only pawn movements) has a subgame value of $\{0,\star \mid \star\}$. Apply the reversibility rule [1], we get $\{0 \mid \star\} = \uparrow$. Therefore, the total subgame value is $\{0 \mid \} + \uparrow = 1\uparrow$.

9. `h3 Kd3`. White is forced to move its *h*–pawn, and in doing so, it looses all the advantage that it had gained in the previous play. File *h* now has the subgame

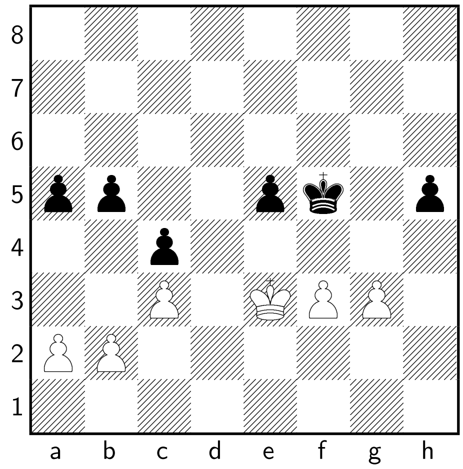

Figure 8: Evans vs. Benko: White to play g4+ and win.

value of $\{\star \mid \star\}$. Black on the other hand patiently moves to its key location of Kd3. Black now just has to sit back and see White push itself to destruction. This subgame value is know as *under* [1, p. 321].

The game described in Figure 7 has a combined combinatorial value of $0 + \star + \{2 \mid \} + \star + \{0 \mid 2\} + 1\uparrow + \textit{under} = 5\textit{under} \approx \textit{under}$. That is, Black has ample enough time to sustain a killing blow to White.

In the actual game Black lost after 1. $\cdots$f3 2. gxf3 gxf3 3. c5 Kd5 4. c6 bxc6 5. b6 axb6 6. a6!.

3.5 Evans vs. Benko

Figure 8 shows the endgame between Evans and Benko played in Las Vegas 1966. The pawn at position g3 is exploited.

1. g4+ hxg4. White's main objective is to clear the board on the King's side since all the other pawns are in a virtual deadlock. It checks the Black King and forces it to capture. The subgame before the check is a second player win. That should not be taken as a disadvantage for White (we will later see why this is true). The subgame has a value of $\{0 \mid 0\} = \star$.

2. fxg4+ Kxg4. White maintains its upper hand and offers another check by capturing pawn on g4. Black is left with no option than to use its King to capture. At this stage the readers would wonder why did White make such a move that makes it loose its near equal board position? This was done in order to draw Black King away from its pawns. The subgame at this stage has a value of $\{ \mid 0\} = -1$. This spare move is due to *e*–pawn.

3. Ke4 Kg3 Now, White will dissolve all the strategically advantage of Black

by treating the *e*–pawn. Black has no option but to move its King down to `Kg3`. The subgame has a value of $\{0 \mid \} = 1$.

4. `Kxe5 Kf3`. The capture of *e*–pawn once again gives White a one spare move. This can easily be deducible from the subgame position after Step 4. Thus, this stage has a subgame value of 1. This one move will result in the ultimate win for White.

5. `Kd5 Ke3`. Similar to the analysis of Step 4, White has a spare move. That is, it can either make a capture as `Kxc4` or move to `Kc5` and force Black to move its pawn.

6. `Kc5 Kd3`. The spare moves of White finally payoff. Black's move can only save its *c*–pawn. Again the subgame has a value of 1.

7. `Kxb5 Kc2`. Both sides converge for pawn cleanup. At this moment, readers would already be feeling, what that one spare move actually means. Black is just too late. The value continues to remain 1.

8. `Kxc4 Kxb2`. White does not go for the cascaded capture of *a*–pawn, rather it takes out *c*–pawn. This is to save at least one of its two pawns. Black captures the *b*–pawn and then threatens the *a*–pawn. This threat makes the subgame value of file *a* as $\{0,\star \mid \star\}$ (applying reversibility we get) $= \{0 \mid \star\} = \uparrow$. File *c* has a value of 1. Thus, the total subgame value is equal to $\{0 \mid \star\} + 1 = 1\uparrow$.

9. `a4 Ka3`. White chooses the en–passant move, which Black follows with a threat. The subgame value of file *a* becomes $\{ \mid 0\} = -1$. Black now gains a one move advantage.

10. `Kb5 Kxc3`. White sacrifices its pawn to threaten *a*–pawn. Files *a* and *b* have a combinatorial value of 1. The pawn simply cannot make a move that is beneficial. Black's capture leaves him far away from the rest of the pieces, and thus, does not affect the analysis. Therefore, the subgame value after the tenth move is 1.

11. `Kxa5`$\cdots$. The capture ensure that the pawn will queen. Black can only wait and pray for mercy. A value of *over* is achieved [1, p. 316].

The complete combinatorial value of the game described in Figure 8 is $\star + -1 + 1 + 1 + 1 + 1 + 1 + 1\uparrow + -1 + 1 + \mathit{over} = 5\mathit{over} \approx \mathit{over}$. That is, White has ample enough time to sustain a killing blow to Black.

The actual game was drawn after 1. `a3? Kf6` 2. `Ke4 Ke6` 3. `Ke3 Kf6`. White intended 4. `g4` (but it must be played with check or Black is not forced to capture) and saw too late it loose to 4. $\cdots$`h4!` 5. `Kf2 Kg5` 6. `Kg2 e4` 7. `fxe4 Kxg4`.

4 The Leftovers

This paper dealt with discrete stepwise combinatorial analysis of chess endgames. In particular five famous endgames recorded in the history of chess games were

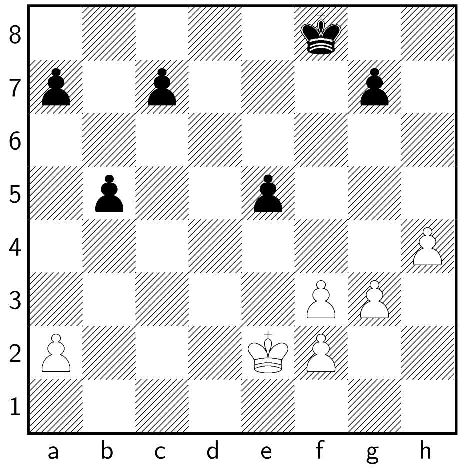

Figure 9: Euwe vs. Alekhine: Black to play ?? and win (exercise for the readers).

analyzed. The proposed analytical technique produced promising results with the same effectiveness as previous work reported on chess endgames.

The readers are invited to exploit the theoretical background presented in this paper to answer the following two questions:

Exercise 1: Analyze the initial board position depicted in Figure 6.

Exercise 2: Project the winner of the game depicted in Figure 9, where Black have the first move. This is an actual endgame played between Euwe and Alekhine (34th match game) in 1935.

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