Union-Find Algorithms

- network connectivity
- quick find
- quick union
- improvements
- applications
Subtext of today’s lecture (and this course)

Steps to developing a usable algorithm.
• Define the problem.
• Find an algorithm to solve it.
• Fast enough?
• If not, figure out why.
• Find a way to address the problem.
• Iterate until satisfied.

The scientific method

Mathematical models and computational complexity

READ Chapter One of Algs in Java
network connectivity
- quick find
- quick union
- improvements
- applications
Network connectivity

Basic abstractions

- set of objects
- union command: connect two objects
- find query: is there a path connecting one object to another?
Union-find applications involve manipulating objects of all types.

- Computers in a network.
- Web pages on the Internet.
- Transistors in a computer chip.
- Variable name aliases.
- Pixels in a digital photo.
- Metallic sites in a composite system.

When programming, convenient to name them 0 to N-1.

- Hide details not relevant to union-find.
- Integers allow quick access to object-related info.
- Could use symbol table to translate from object names.
Union-find abstractions

Simple model captures the essential nature of connectivity.

- **Objects.**
  
<table>
<thead>
<tr>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>grid points</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

- **Disjoint sets of objects.**
  
<table>
<thead>
<tr>
<th>0</th>
<th>1</th>
<th>{ 2 3 9 }</th>
<th>{ 5 6 }</th>
<th>7</th>
<th>{ 4 8 }</th>
</tr>
</thead>
<tbody>
<tr>
<td>subsets of connected grid points</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

- **Find** query: are objects 2 and 9 in the same set?
  
<table>
<thead>
<tr>
<th>0</th>
<th>1</th>
<th>{ 2 3 9 }</th>
<th>{ 5–6 }</th>
<th>7</th>
<th>{ 4–8 }</th>
</tr>
</thead>
<tbody>
<tr>
<td>are two grid points connected?</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

- **Union** command: merge sets containing 3 and 8.
  
<table>
<thead>
<tr>
<th>0</th>
<th>1</th>
<th>{ 2 3 4 8 9 }</th>
<th>{ 5–6 }</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>add a connection between two grid points</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Connected components

Connected component: set of mutually connected vertices

Each union command reduces by 1 the number of components

<table>
<thead>
<tr>
<th>in</th>
<th>out</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>4</td>
<td>9</td>
</tr>
<tr>
<td>8</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>5</td>
<td>6</td>
</tr>
<tr>
<td>2</td>
<td>9</td>
</tr>
<tr>
<td>5</td>
<td>9</td>
</tr>
<tr>
<td>7</td>
<td>3</td>
</tr>
</tbody>
</table>

3 = 10 - 7 components

7 union commands
Network connectivity: larger example

\[ \text{find}(u, v) \]
Network connectivity: larger example

\textit{find}(u, v) \ ?

\textbf{true}

63 components
Union-find abstractions

- Objects.
- Disjoint sets of objects.
- **Find queries:** are two objects in the same set?
- **Union commands:** replace sets containing two items by their union

**Goal.** Design efficient data structure for union-find.

- Find queries and union commands may be intermixed.
- Number of operations \( M \) can be huge.
- Number of objects \( N \) can be huge.
network connectivity
quick find
quick union
improvements
applications
Quick-find \textit{[eager approach]}

Data structure.
- Integer array $id[]$ of size $N$.
- Interpretation: $p$ and $q$ are connected if they have the same $id$.

\[
\begin{array}{cccccccccc}
 i & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
id[i] & 0 & 1 & 9 & 9 & 9 & 6 & 6 & 7 & 8 & 9 \\
\end{array}
\]

5 and 6 are connected
2, 3, 4, and 9 are connected
Quick-find  [eager approach]

Data structure.
• Integer array \( id[] \) of size \( N \).
• Interpretation: \( p \) and \( q \) are connected if they have the same id.

Find. Check if \( p \) and \( q \) have the same id.

Union. To merge components containing \( p \) and \( q \), change all entries with \( id[p] \) to \( id[q] \).

<table>
<thead>
<tr>
<th>i</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>id[i]</td>
<td>0</td>
<td>1</td>
<td>9</td>
<td>9</td>
<td>9</td>
<td>6</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>9</td>
</tr>
</tbody>
</table>

5 and 6 are connected
2, 3, 4, and 9 are connected

<table>
<thead>
<tr>
<th>i</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>id[i]</td>
<td>0</td>
<td>1</td>
<td>6</td>
<td>6</td>
<td>6</td>
<td>6</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>6</td>
</tr>
</tbody>
</table>

3 and 6 not connected

union of 3 and 6
2, 3, 4, 5, 6, and 9 are connected

problem: many values can change
Quick-find example

\[
\begin{align*}
3-4 & \quad 0 \quad 1 \quad 2 \quad 4 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9 \\
4-9 & \quad 0 \quad 1 \quad 2 \quad 9 \quad 9 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9 \\
8-0 & \quad 0 \quad 1 \quad 2 \quad 9 \quad 9 \quad 5 \quad 6 \quad 7 \quad 0 \quad 9 \\
2-3 & \quad 0 \quad 1 \quad 9 \quad 9 \quad 9 \quad 5 \quad 6 \quad 7 \quad 0 \quad 9 \\
5-6 & \quad 0 \quad 1 \quad 9 \quad 9 \quad 9 \quad 6 \quad 6 \quad 7 \quad 0 \quad 9 \\
5-9 & \quad 0 \quad 1 \quad 9 \quad 9 \quad 9 \quad 9 \quad 9 \quad 7 \quad 0 \quad 9 \\
7-3 & \quad 0 \quad 1 \quad 9 \quad 9 \quad 9 \quad 9 \quad 9 \quad 9 \quad 0 \quad 9 \\
4-8 & \quad 0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \\
6-1 & \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1
\end{align*}
\]

Problem: many values can change
public class QuickFind
{
    private int[] id;

    public QuickFind(int N)
    {
        id = new int[N];
        for (int i = 0; i < N; i++)
        {
            id[i] = i;
        }
    }

    public boolean find(int p, int q)
    {
        return id[p] == id[q];
    }

    public void unite(int p, int q)
    {
        int pid = id[p];
        for (int i = 0; i < id.length; i++)
        {
            if (id[i] == pid) id[i] = id[q];
        }
    }
}
Quick-find is too slow

Quick-find algorithm may take $\sim MN$ steps to process $M$ union commands on $N$ objects.

Rough standard (for now).
- $10^9$ operations per second.
- $10^9$ words of main memory.
- Touch all words in approximately 1 second.

Ex. Huge problem for quick-find.
- $10^{10}$ edges connecting $10^9$ nodes.
- Quick-find takes more than $10^{19}$ operations.
- $300+$ years of computer time!

Paradoxically, quadratic algorithms get worse with newer equipment.
- New computer may be 10x as fast.
- But, has 10x as much memory so problem may be 10x bigger.
- With quadratic algorithm, takes 10x as long!

a truism (roughly) since 1950!
network connectivity
quick find
quick union
improvements
applications
Quick-union [lazy approach]

Data structure.
- Integer array \( id[] \) of size \( N \).
- Interpretation: \( id[i] \) is parent of \( i \).
- Root of \( i \) is \( id[id[id[...id[i]...]]] \).

<table>
<thead>
<tr>
<th>i</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>( id[i] )</td>
<td>0</td>
<td>1</td>
<td>9</td>
<td>4</td>
<td>9</td>
<td>6</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>9</td>
</tr>
</tbody>
</table>

```
0   1   9   6   7   8
   2   4   5
     3
```

3's root is 9; 5's root is 6
Quick-union [lazy approach]

Data structure.
• Integer array `id[]` of size `N`.
• Interpretation: `id[i]` is parent of `i`.
• Root of `i` is `id[id[id[...id[i]...]]]`.

Find. Check if `p` and `q` have the same root.

```
   i  0  1  2  3  4  5  6  7  8  9
   id[i] 0  1  9  4  9  6  6  7  8  9
```

keep going until it doesn't change

3’s root is 9; 5’s root is 6
3 and 5 are not connected
Quick-union  [lazy approach]

Data structure.
• Integer array \( id[] \) of size \( N \).
• Interpretation: \( id[i] \) is parent of \( i \).
• Root of \( i \) is \( id[id[id[...id[i]...]]] \).

\[
\begin{array}{cccccccccc}
i & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
id[i] & 0 & 1 & 9 & 4 & 9 & 6 & 6 & 7 & 8 & 9 \\
\end{array}
\]

Find. Check if \( p \) and \( q \) have the same root.

Union. To merge sets containing \( p \) and \( q \), set the id of \( p \)'s root to the id of \( q \)'s root.

\[
\begin{array}{cccccccccc}
i & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
id[i] & 0 & 1 & 9 & 4 & 9 & 6 & 6 & 7 & 8 & 6 \\
\end{array}
\]
### Quick-union example

<table>
<thead>
<tr>
<th>Operation</th>
<th>Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>3-4</td>
<td>0 1 2 4 4 5 6 7 8 9</td>
</tr>
<tr>
<td>4-9</td>
<td>0 1 2 4 9 5 6 7 8 9</td>
</tr>
<tr>
<td>8-0</td>
<td>0 1 2 4 9 5 6 7 0 9</td>
</tr>
<tr>
<td>2-3</td>
<td>0 1 9 4 9 5 6 7 0 9</td>
</tr>
<tr>
<td>5-6</td>
<td>0 1 9 4 9 6 6 7 0 9</td>
</tr>
<tr>
<td>5-9</td>
<td>0 1 9 4 9 6 9 7 0 9</td>
</tr>
<tr>
<td>7-3</td>
<td>0 1 9 4 9 6 9 9 0 9</td>
</tr>
<tr>
<td>4-8</td>
<td>0 1 9 4 9 6 9 9 0 0</td>
</tr>
<tr>
<td>6-1</td>
<td>1 1 9 4 9 6 9 9 0 0</td>
</tr>
</tbody>
</table>

Problem: trees can get tall
Quick-union: Java implementation

```java
public class QuickUnion {
    private int[] id;

    public QuickUnion(int N) {
        id = new int[N];
        for (int i = 0; i < N; i++) id[i] = i;
    }

    private int root(int i) {
        while (i != id[i]) i = id[i];
        return i;
    }

    public boolean find(int p, int q) {
        return root(p) == root(q);
    }

    public void unite(int p, int q) {
        int i = root(p);
        int j = root(q);
        id[i] = j;
    }
}
```
Quick-union is also too slow

**Quick-find defect.**
- Union too expensive (N steps).
- Trees are flat, but too expensive to keep them flat.

**Quick-union defect.**
- Trees can get tall.
- Find too expensive (could be N steps)
- Need to do find to do union

<table>
<thead>
<tr>
<th>algorithm</th>
<th>union</th>
<th>find</th>
</tr>
</thead>
<tbody>
<tr>
<td>Quick-find</td>
<td>N</td>
<td>1</td>
</tr>
<tr>
<td>Quick-union</td>
<td>N*</td>
<td>N</td>
</tr>
</tbody>
</table>

* includes cost of find

*worst case*
network connectivity
quick find
quick union
improvements
applications
Improvement 1: Weighting

Weighted quick-union.
- Modify quick-union to avoid tall trees.
- Keep track of size of each component.
- Balance by linking small tree below large one.

Ex. Union of 5 and 3.
- Quick union: link 9 to 6.
- Weighted quick union: link 6 to 9.

![Diagram](image)
Weighted quick-union example

3–4  0 1 2 3 3 5 6 7 8 9
4–9  0 1 2 3 3 5 6 7 8 3
8–0  8 1 2 3 3 5 6 7 8 3
2–3  8 1 3 3 3 5 6 7 8 3
5–6  8 1 3 3 3 5 5 7 8 3
5–9  8 1 3 3 3 3 5 7 8 3
7–3  8 1 3 3 3 3 5 3 8 3
4–8  8 1 3 3 3 3 5 3 3 3
6–1  8 3 3 3 3 3 5 3 3 3

no problem: trees stay flat
Weighted quick-union: Java implementation

Java implementation.
• Almost identical to quick-union.
• Maintain extra array sz[] to count number of elements in the tree rooted at i.

Find. Identical to quick-union.

Union. Modify quick-union to
• merge smaller tree into larger tree
• update the sz[] array.

```java
if (sz[i] < sz[j]) { id[i] = j; sz[j] += sz[i]; }
else sz[i] < sz[j] { id[j] = i; sz[i] += sz[j]; }
```
Weighted quick-union analysis

Analysis.
• Find: takes time proportional to depth of \( p \) and \( q \).
• Union: takes constant time, given roots.
• Fact: depth is at most \( \log N \). [needs proof]

<table>
<thead>
<tr>
<th>Data Structure</th>
<th>Union</th>
<th>Find</th>
</tr>
</thead>
<tbody>
<tr>
<td>Quick-find</td>
<td>( N )</td>
<td>( 1 )</td>
</tr>
<tr>
<td>Quick-union</td>
<td>( N^* )</td>
<td>( N )</td>
</tr>
<tr>
<td>Weighted QU</td>
<td>( \log N^* )</td>
<td>( \log N )</td>
</tr>
</tbody>
</table>

* includes cost of find

Stop at guaranteed acceptable performance? No, easy to improve further.
**Improvement 2: Path compression**

Path compression. Just after computing the root of $i$, set the $id$ of each examined node to $\text{root}(i)$.
Path compression.

- Standard implementation: add second loop to `root()` to set the id of each examined node to the root.
- Simpler one-pass variant: make every other node in path point to its grandparent.

```java
public int root(int i) {
    while (i != id[i]) {
        id[i] = id[id[i]]; // only one extra line of code!
        i = id[i];
    }
    return i;
}
```

In practice. No reason not to! Keeps tree almost completely flat.
Weighted quick-union with path compression

3–4  0 1 2 3 3 5 6 7 8 9
4–9  0 1 2 3 3 5 6 7 8 3
8–0  8 1 2 3 3 5 6 7 8 3
2–3  8 1 3 3 3 5 6 7 8 3
5–6  8 1 3 3 3 5 5 7 8 3
5–9  8 1 3 3 3 3 5 7 8 3
7–3  8 1 3 3 3 3 5 3 8 3
4–8  8 1 3 3 3 3 5 3 3 3
6–1  8 3 3 3 3 3 3 3 3

no problem: trees stay VERY flat
WQUPC performance

**Theorem.** Starting from an empty data structure, any sequence of M union and find operations on N objects takes $O(N + M \ lg^* N)$ time.
- Proof is very difficult.
- But the algorithm is still simple!

**Linear algorithm?**
- Cost within constant factor of reading in the data.
- In theory, WQUPC is not quite linear.
- In practice, WQUPC is linear.

**Amazing fact:**
- In theory, no linear linking strategy exists.

<table>
<thead>
<tr>
<th>N</th>
<th>lg* N</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>16</td>
<td>3</td>
</tr>
<tr>
<td>65536</td>
<td>4</td>
</tr>
<tr>
<td>$2^{65536}$</td>
<td>5</td>
</tr>
</tbody>
</table>
Summary

**Bottom line.** WQUPC makes it possible to solve problems that could not otherwise be addressed.

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>Worst-case Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>quick-find</td>
<td>$M \cdot N$</td>
</tr>
<tr>
<td>quick-union</td>
<td>$M \cdot N$</td>
</tr>
<tr>
<td>weighted QU</td>
<td>$N + M \log N$</td>
</tr>
<tr>
<td>QU + path compression</td>
<td>$N + M \log N$</td>
</tr>
<tr>
<td>weighted QU + path compression</td>
<td>$N + M \lg^* N$</td>
</tr>
</tbody>
</table>

*M union-find operations on a set of N objects*

**Ex.** [$10^9$ unions and finds with $10^9$ objects]

- WQUPC reduces time from 30 years to 6 seconds.
- Supercomputer won't help much; good algorithm enables solution.
network connectivity
quick find
quick union
improvements
applications
Union-find applications

✓ Network connectivity.
  • Percolation.
  • Image processing.
  • Least common ancestor.
  • Equivalence of finite state automata.
  • Hinley-Milner polymorphic type inference.
  • Kruskal’s minimum spanning tree algorithm.
  • Games (Go, Hex)
  • Compiling equivalence statements in Fortran.
Percolation

A model for many physical systems
- N-by-N grid.
- Each square is vacant or occupied.
- Grid **percolates** if top and bottom are connected by vacant squares.

<table>
<thead>
<tr>
<th>model</th>
<th>system</th>
<th>vacant site</th>
<th>occupied site</th>
<th>percolates</th>
</tr>
</thead>
<tbody>
<tr>
<td>electricity</td>
<td>material</td>
<td>conductor</td>
<td>insulated</td>
<td>conducts</td>
</tr>
<tr>
<td>fluid flow</td>
<td>material</td>
<td>empty</td>
<td>blocked</td>
<td>porous</td>
</tr>
<tr>
<td>social interaction</td>
<td>population</td>
<td>person</td>
<td>empty</td>
<td>communicates</td>
</tr>
</tbody>
</table>
Percolation phase transition

Likelihood of percolation depends on site vacancy probability $p$

$\text{Experiments show a threshold } p^*\text{:}$
- $p > p^*$: almost certainly percolates
- $p < p^*$: almost certainly does not percolate

Q. What is the value of $p^*$?
UF solution to find percolation threshold

- Initialize whole grid to be “not vacant”
- Implement “make site vacant” operation that does `union()` with adjacent sites
- Make all sites on top and bottom rows vacant
- Make random sites vacant until `find(top, bottom)`
- Vacancy percentage estimates $p^*$
Percolation

Q. What is percolation threshold $p^*$?
A. about 0.592746 for large square lattices.

percolation constant known only via simulation

percolates does not percolate
Hex

**Hex.** [Piet Hein 1942, John Nash 1948, Parker Brothers 1962]
- Two players alternate in picking a cell in a hex grid.
- Black: make a black path from upper left to lower right.
- White: make a white path from lower left to upper right.


**Union-find application.** Algorithm to detect when a player has won.
Subtext of today’s lecture (and this course)

Steps to developing an usable algorithm.
- Define the problem.
- Find an algorithm to solve it.
- Fast enough?
- If not, figure out why.
- Find a way to address the problem.
- Iterate until satisfied.

The scientific method

Mathematical models and computational complexity

READ Chapter One of Algs in Java