CSE 2320 Notes 3: Summations

(Last updated 8/17/06 6:39 AM)

CLRS, Appendix A

GEOMETRIC SERIES (review)

$$\sum_{k=0}^{t} x^k = \frac{x^{t+1}-1}{x-1} \qquad x \neq 1 \qquad \text{[Not hard to verify by math induction]}$$

$$\sum_{k=0}^{t} x^k \leq \sum_{k=0}^{\infty} x^k = \lim_{k \to \infty} \frac{x^k - 1}{x-1} = \frac{1}{1-x} \qquad 0 < x < 1$$

HARMONIC SERIES

$$\ln n \leq H_n = \sum_{k=1}^{n} \frac{1}{k} \leq \ln n + .577 \leq \ln n + 1 \qquad \text{[Not hard to verify using integrals]}$$

As n approaches ∞, $\frac{H_n}{H_{2n}}$ approaches?

A. 1 B. 2 C. $\ln n$ D. $n!$

Problem A. 1-2 Show that

$$\sum_{k=1}^{n} \frac{1}{2k-1} \leq \ln\left(\sqrt{n}\right) + O(1)$$

Consider $n = 6$: $\frac{1}{1}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}+\frac{1}{11} = \frac{1}{1}+\frac{1}{2}+\cdots+\frac{1}{12} - \frac{1}{2}\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}\right)$

$$= H_{2n} - \frac{1}{2} H_n$$

$$\leq \ln 2n + O(1) - \frac{1}{2}\ln n = \ln 2 + \ln n + O(1) - \ln\sqrt{n}$$

$$= \ln n - \ln\sqrt{n} + \ln 2 + O(1)$$

$$= \ln\left(\frac{n}{\sqrt{n}}\right) + O(1) = \ln\left(\sqrt{n}\right) + O(1)$$

BOUNDING SUMMATIONS USING MATH INDUCTION AND INEQUALITIES

[Techniques are especially important for recurrences in notes 4]

Show $\sum_{i=1}^{n} i^2 = \Theta\left(n^3\right)$ [Trivial to show using integration. $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$]

a. Show $O(n^3)$

(i) $\sum_{i=1}^{n=1} i^2 = 1 \le cn^3$ using any constant $c \ge 1$

(ii) Suppose this holds for n:

$$\sum_{i=1}^{n} i^2 \le cn^3$$

Now go on to $n + 1$ and show that the bound *still holds*

$$\sum_{i=1}^{n+1} i^2 = \sum_{i=1}^{n} i^2 + (n+1)^2$$

$$= \sum_{i=1}^{n} i^2 + n^2 + 2n + 1$$

$$\le cn^3 + n^2 + 2n + 1$$

$$= ???$$

$$\le c(n+1)^3$$

The bridging step (???) separates the bounding term ($c(n+1)^3$) from everything else (x):

$$c(n+1)^3 + x = cn^3 + n^2 + 2n + 1$$

$$x = cn^3 + n^2 + 2n + 1 - cn^3 - 3cn^2 - 3cn - c = (1-3c)n^2 + (2-3c)n + 1 - c$$

So ??? is now $c(n+1)^3 + \left[(1-3c)n^2 + (2-3c)n + 1 - c\right]$

Can drop [. . .] (through $\le$) if it cannot become positive. Happens if $c \ge 1$

b. Show $\Omega(n^3)$

(i) $\sum_{i=1}^{n=1} i^2 = 1 \geq cn^3$ using any constant $0 < c \leq 1$

(ii) Suppose this holds for n:

$$\sum_{i=1}^{n} i^2 \geq cn^3$$

Now go on to $n + 1$ and show that the bound *still holds*

$$\begin{aligned}\sum_{i=1}^{n+1} i^2 &= \sum_{i=1}^{n} i^2 + (n+1)^2 \\ &= \sum_{i=1}^{n} i^2 + n^2 + 2n + 1 \\ &\geq cn^3 + n^2 + 2n + 1 \\ &= ??? \\ &\geq c(n+1)^3\end{aligned}$$

The bridging step (???) involves the same algebra as before.

Can drop [. . .] (through ≥) if it cannot become negative. Happens if $0 < c \leq 1/3$

Suppose we attempt to show $\sum_{i=1}^{n} i^2 = \Theta(n^2)$

a. Show $O(n^2)$

(i) $\sum_{i=1}^{n=1} i^2 = 1 \leq cn^2$ using any constant $c \geq 1$

(ii) Suppose this holds for n:

$$\sum_{i=1}^{n} i^2 \leq cn^2$$

Now attempt to go on to $n + 1$.

$$\sum_{i=1}^{n+1} i^2 = \sum_{i=1}^{n} i^2 + (n+1)^2$$

$$= \sum_{i=1}^{n} i^2 + n^2 + 2n + 1$$

$$\leq cn^2 + n^2 + 2n + 1$$

$$= ???$$

$$\leq c(n+1)^2$$

The bridging step separates the bounding term from everything else:

$$c(n+1)^2 + x = cn^2 + n^2 + 2n + 1$$

$$x = cn^2 + n^2 + 2n + 1 - cn^2 - 2cn - c = n^2 + (2 - 2c)n + 1 - c$$

So ??? is now $c(n+1)^2 + \left[n^2 + (2-2c)n + 1 - c\right]$

Can drop [. . .] (through ≤) if it cannot become positive. ***Fails as n grows.***

b. Can still show $\Omega(n^2)$

(i) $\sum_{i=1}^{n=1} i^2 = 1 \geq cn^2$ using any constant $0 < c \leq 1$

(ii) Suppose this holds for n:

$$\sum_{i=1}^{n} i^2 \geq cn^2$$

Now go on to $n + 1$.

$$\sum_{i=1}^{n+1} i^2 = \sum_{i=1}^{n} i^2 + (n+1)^2$$

$$= \sum_{i=1}^{n} i^2 + n^2 + 2n + 1$$

$$\geq cn^2 + n^2 + 2n + 1$$

$$= ???$$

$$\geq c(n+1)^2$$

The bridging step separates the bounding term from everything else:

$$c(n+1)^2 + x = cn^2 + n^2 + 2n + 1$$

$$x = cn^2 + n^2 + 2n + 1 - cn^2 - 2cn - c = n^2 + (2-2c)n + 1 - c$$

So ??? is now $c(n+1)^2 + \left[n^2 + (2-2c)n + 1 - c\right]$

Can drop [. . .] (through ≥) if it cannot become negative.

Happens if $0 < c \leq 1$ (or as n grows).

APPROXIMATION BY INTEGRALS

For a monotonically increasing function ($x \leq y \Rightarrow f(x) \leq f(y)$):

$$\int_{m-1}^{n} f(x)dx \leq \sum_{k=m}^{n} f(k) \leq \int_{m}^{n+1} f(x)dx$$

p. 1068 for picture proof