CSE 2320 Notes 4: Recurrences

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CLRS, 4.1-4.2

BOUNDING RECURRENCES ASYMPTOTICALLY

Goal: Take a function $T(n)$ that is defined recursively and find $f(n)$ such that $T(n) \in \Theta(f(n))$.

Need to establish both $T(n) \in O(f(n))$ and $T(n) \in \Omega(f(n))$.

Use the limit theorems? Must establish constants (c, n_0) for both O and Ω definitions.

But, it is convenient to have $n_0 = 1$ in both cases and show that a c exists for each of the bounds.

RECURRENCES AND CONSTANTS

Consider the following recurrence:

$$T(n) = T\left(\frac{n}{2}\right) + e = T\left(\frac{n}{2}\right) + \Theta(1)$$
$$T(1) = d = \Theta(1)$$

Suppose $n = 2^k$:

$$T\left(2^k\right) = T\left(2^{k-1}\right) + e = T\left(2^{k-2}\right) + 2e = T\left(2^{k-3}\right) + 3e = \cdots = T\left(2^0\right) + ek$$
$$= d + ek = d + e \lg n = \Theta(\log n)$$

Consider the following recurrence:

$$T(n) = 2T\left(\frac{n}{2}\right) + en = 2T\left(\frac{n}{2}\right) + \Theta(n)$$
$$T(1) = d = \Theta(1)$$

Suppose $n = 2^k$:

$$T\left(2^k\right) = 2T\left(2^{k-1}\right) + en$$

$$= 2\left(2T\left(2^{k-2}\right) + e2^{k-1}\right) + en = 4T\left(2^{k-2}\right) + e2^k + n = 4T\left(2^{k-2}\right) + 2en$$

$$= 4\left(2T\left(2^{k-3}\right) + e2^{k-2}\right) + 2en = 8T\left(2^{k-3}\right) + e2^k + 2en = 8T\left(2^{k-3}\right) + 3en$$

$$= \cdots$$

$$= nT(2^0) + ekn = (d + ek)n = (d + e\lg n)n = \Theta(n \log n)$$

Constant does not usually matter, except:

$$T(n) = T\left(\frac{n}{2}\right)^2$$

$$T(1) = 2 = \Theta(1)$$

$$T(8) = T(4)^2 = \left[T(2)^2\right]^2 = T(1)^{2^{2^2}} = 2^{2^{2^2}} = 256 = 2^n$$

Now suppose $T(1) = 3 = \Theta(1)$

$$T(8) = 3^{2^{2^2}} = 6561 = 3^n$$

THE SUBSTITUTION METHOD FOR BOUNDING RECURRENCES

Method

Guess bound (lower and/or upper) [On exams the guess will be given to you]

Verify by math induction (solve for constants)

i) Assume bounding hypothesis works for $k < n$

ii) Show bounding hypothesis works for n in ***exactly*** the same form as (i).

Refine bound

Example: Binary search recurrence (number of probes)

$$T(n) = T\left(\frac{n}{2}\right) + 1$$

$$\mathrm{O}(\log n)$$

Assume $T(k) \le c \lg k$ for $k < n$

$$T\left(\frac{n}{2}\right) \le c \lg\left(\frac{n}{2}\right) = c(\lg n - \lg 2) = c \lg n - c$$

$$T(n) = T\left(\frac{n}{2}\right) + 1$$

$$\le c \lg n - c + 1$$

$$\le c \lg n \text{ if } c \ge 1$$

$$\Omega(\log n)$$

Assume $T(k) \ge c \lg k$ for $k < n$

$$T\left(\frac{n}{2}\right) \ge c \lg\left(\frac{n}{2}\right) = c(\lg n - \lg 2) = c \lg n - c$$

$$T(n) = T\left(\frac{n}{2}\right) + 1$$

$$\ge c \lg n - c + 1$$

$$\ge c \lg n \text{ if } 0 < c \le 1$$

$$T(n) = 2T\left(\frac{n}{2}\right) + dn$$

$$\mathrm{O}(n \log n)$$

Assume $T(k) \le ck \lg k$ for $k < n$

$$T\left(\frac{n}{2}\right) \le c\frac{n}{2} \lg\left(\frac{n}{2}\right) = c\frac{n}{2}(\lg n - \lg 2) = c\frac{n}{2} \lg n - c\frac{n}{2}$$

$$T(n) = 2T\left(\frac{n}{2}\right) + dn$$

$$\le 2\left(c\frac{n}{2} \lg n - c\frac{n}{2}\right) + dn$$

$$= cn \lg n - cn + dn$$

$$\le cn \lg n \text{ if } c \ge d$$

$\Omega(n \log n)$

Assume $T(k) \ge ck \lg k$ for $k < n$

$$T\left(\frac{n}{2}\right) \ge c\,\frac{n}{2}\lg\left(\frac{n}{2}\right) = c\,\frac{n}{2}(\lg n - \lg 2) = c\,\frac{n}{2}\lg n - c\,\frac{n}{2}$$

$$\begin{aligned} T(n) &= 2T\left(\frac{n}{2}\right) + dn \\ &\ge 2\left(c\,\frac{n}{2}\lg n - c\,\frac{n}{2}\right) + dn \\ &= cn\lg n - cn + dn \\ &\ge cn\lg n \text{ if } 0 < c \le d \end{aligned}$$

Exercise 4.3-1a

$$T(n) = 4T\left(\frac{n}{2}\right) + n$$

Try $O\left(n^3\right)$ and confirm by math induction:

Assume $T(k) \le ck^3$ for $k < n$

$$T\left(\frac{n}{2}\right) \le c\,\frac{n^3}{8}$$

$$\begin{aligned} T(n) &= 4T\left(\frac{n}{2}\right) + n \\ &\le 4c\,\frac{n^3}{8} + n \\ &= \frac{c}{2}n^3 + n \\ &= cn^3 - \frac{c}{2}n^3 + n \qquad -\frac{c}{2}n^3 + n \le 0 \text{ if } c \ge 2 \\ &\le cn^3 \end{aligned}$$

Improve bound to $O\left(n^2\right)$ and confirm:

Assume $T(k) \le ck^2$ for $k < n$

$$T\left(\frac{n}{2}\right) \le c\frac{n^2}{4}$$

$$T(n) = 4T\left(\frac{n}{2}\right) + n \le 4c\frac{n^2}{4} + n = cn^2 + n \text{ STUCK!}$$

$\Omega\left(n^3\right)$ as lower bound:

Assume $T(k) \ge ck^3$ for $k < n$

$$T\left(\frac{n}{2}\right) \ge c\frac{n^3}{8}$$

$$T(n) = 4T\left(\frac{n}{2}\right) + n$$

$$\ge 4c\frac{n^3}{8} + n$$

$$= \frac{c}{2}n^3 + n$$

$$= cn^3 - \frac{c}{2}n^3 + n \qquad -\frac{c}{2}n^3 + n \ge 0\,?$$

$$\ge cn^3 \text{ DID NOT PROVE!!!}$$

$\Omega\left(n^2\right)$ as lower bound:

Assume $T(k) \ge ck^2$ for $k < n$

$$T\left(\frac{n}{2}\right) \ge c\frac{n^2}{4}$$

$$T(n) = 4T\left(\frac{n}{2}\right) + n \ge 4c\frac{n^2}{4} + n = cn^2 + n \ge cn^2 \text{ for } 0 < c$$

What's going on . . .

$$T(1) = d$$
$$T(2) = 4T(1) + 2 = 4d + 2$$
$$T(4) = 4T(2) + 4 = 4(4d + 2) + 4 = 16d + 8 + 4 = 16d + 12$$
$$T(8) = 4T(4) + 8 = 4(16d + 12) + 8 = 64d + 48 + 8 = 64d + 56$$
$$T(16) = 4T(8) + 16 = 4(64d + 56) + 16 = 256d + 224 + 16 = 256d + 240$$

Hypothesis: $T(n) = dn^2 + n^2 - n = (d+1)n^2 - n = cn^2 - n$

$O(n^2)$:

Assume $T(k) \le ck^2 - k$ for $k < n$

$$T\left(\frac{n}{2}\right) \le c\frac{n}{4}^2 - \frac{n}{2}$$
$$T(n) = 4T\left(\frac{n}{2}\right) + n \le 4\left(c\frac{n}{4}^2 - \frac{n}{2}\right) + n = cn^2 - 2n + n = cn^2 - n$$

$\Omega(n^2)$: [This bound was already proven.]

Assume $T(k) \ge ck^2 - k$ for $k < n$

$$T\left(\frac{n}{2}\right) \ge c\frac{n}{4}^2 - \frac{n}{2}$$
$$T(n) = 4T\left(\frac{n}{2}\right) + n \ge 4\left(c\frac{n}{4}^2 - \frac{n}{2}\right) + n = cn^2 - 2n + n = cn^2 - n$$

Exercise 4-1.a

$$T(n) = 2T\left(\frac{n}{2}\right) + n^3$$

$\mathrm{O}\left(n^3 \log n\right)$:

Assume $T(k) \le ck^3 \lg k$ for $k < n$

$$T\left(\frac{n}{2}\right) \le c\frac{n^3}{8}\lg\frac{n}{2} = c\frac{n^3}{8}\lg n - c\frac{n^3}{8}$$

$$T(n) = 2T\left(\frac{n}{2}\right) + n^3 \le 2\left(c\frac{n^3}{8}\lg n - c\frac{n^3}{8}\right) + n^3 = c\frac{n^3}{4}\lg n - c\frac{n^3}{4} + n^3$$

$$= cn^3\lg n - \frac{3}{4}cn^3\lg n - c\frac{n^3}{4} + n^3$$

$$\le cn^3\lg n \text{ if } c \ge \frac{4}{3} \text{ (or } n \text{ is "sufficiently large", in the sense of } n_0\text{)}$$

$\mathrm{O}\left(n^3\right)$:

Assume $T(k) \le ck^3$ for $k < n$

$$T\left(\frac{n}{2}\right) \le c\frac{n^3}{8}$$

$$T(n) = 2T\left(\frac{n}{2}\right) + n^3 \le 2c\frac{n^3}{8} + n^3 = c\frac{n^3}{4} + n^3$$

$$= cn^3 - \frac{3}{4}cn^3 + n^3$$

$$\le cn^3 \text{ if } c \ge \frac{4}{3}$$

$\Omega\left(n^3\right)$

Assume $T(k) \ge ck^3$ for $k < n$

$$T\left(\frac{n}{2}\right) \ge c\frac{n^3}{8}$$

$$T(n) = 2T\left(\frac{n}{2}\right) + n^3 \ge 2c\frac{n^3}{8} + n^3 = c\frac{n^3}{4} + n^3$$

$$= cn^3 - \frac{3}{4}cn^3 + n^3$$

$$\ge cn^3 \text{ if } 0 < c \le \frac{4}{3}$$

Exercise 4.1 h

$$T(n) = T\left(\sqrt{n}\right) + 1$$

$O(\log n)$:

Assume $T(k) \le c \lg k$ for $k < n$

$$T\left(\sqrt{n}\right) \le c \lg \sqrt{n} = c\frac{\lg n}{2}$$

$$T(n) = T\left(\sqrt{n}\right) + 1 \le c\frac{\lg n}{2} + 1 = c \lg n - c\frac{\lg n}{2} + 1$$

$$\le c \lg n \quad \text{if } c \ge 2$$

$O(\log\log n)$

Assume $T(k) \le c \lg\lg k$ for $k < n$

$$T\left(\sqrt{n}\right) \le c \lg\lg \sqrt{n} = c \lg \frac{\lg n}{2} = c \lg\lg n - c$$

$$T(n) = T\left(\sqrt{n}\right) + 1 \le c \lg\lg n - c + 1$$

$$\le c \lg\lg n \quad \text{if } c \ge 1$$

$\Omega(\log\log n)$

Assume $T(k) \ge c \lg\lg k$ for $k < n$

$$T\left(\sqrt{n}\right) \ge c \lg\lg \sqrt{n} = c \lg \frac{\lg n}{2} = c \lg\lg n - c$$

$$T(n) = T\left(\sqrt{n}\right) + 1 \ge c \lg\lg n - c + 1$$

$$\ge c \lg\lg n \quad \text{if } 0 < c \le 1$$

Example:

$$L(n) = 2L\left(\frac{n}{2}\right) + \lg n \text{ is } \Theta(n)$$

$O(n)$:

$$L(k) \le ck \text{ for } k < n$$
$$L\left(\frac{n}{2}\right) \le c\frac{n}{2}$$
$$L(n) = 2L\left(\frac{n}{2}\right) + \lg n \le 2c\frac{n}{2} + \lg n = cn + \lg n \text{ stuck!}$$

$\Omega(n)$:

$$L(k) \ge ck \text{ for } k < n$$
$$L\left(\frac{n}{2}\right) \ge c\frac{n}{2}$$
$$L(n) = 2L\left(\frac{n}{2}\right) + \lg n \ge 2c\frac{n}{2} + \lg n = cn + \lg n \ge cn$$

Examine a few cases:

$$L(1) = d$$
$$L(2) = 2L(1) + 1 = 2d + 1$$
$$L(4) = 2L(2) + 2 = 2(2d + 1) + 2 = 4d + 2 + 2 = 4d + 4$$
$$L(8) = 2L(4) + 3 = 2(4d + 4) + 3 = 8d + 8 + 3 = 8d + 11$$
$$L(16) = 2L(8) + 4 = 2(8d + 11) + 4 = 16d + 22 + 4 = 16d + 26$$
$$L(32) = 2L(16) + 5 = 2(16d + 26) + 5 = 32d + 52 + 5 = 32d + 57$$
$$L(64) = 2L(32) + 6 = 2(32d + 57) + 6 = 64d + 114 + 6 = 64d + 120$$

$$\text{Hypothesis: } L(n) = nd + 2n - \lg n - 2 = (d + 2)n - \lg n - 2 \le cn - \lg n - 2$$

New try at $O(n)$:

$$L(k) \le ck - \lg k - 2 \text{ for } k < n$$
$$L\left(\frac{n}{2}\right) \le c\frac{n}{2} - \lg\frac{n}{2} - 2$$
$$= c\frac{n}{2} - \lg n + 1 - 2 = c\frac{n}{2} - \lg n - 1$$
$$L(n) = 2L\left(\frac{n}{2}\right) + \lg n \le 2\left(c\frac{n}{2} - \lg n - 1\right) + \lg n$$
$$= cn - 2\lg n - 2 + \lg n = cn - \lg n - 2$$

Example:

$$T(n) = 3T\left(\frac{n}{3}\right) + 2$$

$\mathrm{O}(n)$:

Assume $T(k) \le ck$ for $k < n$

$$T\left(\frac{n}{3}\right) \le \frac{cn}{3}$$

$$T(n) = 3T\left(\frac{n}{3}\right) + 2 \le 3\frac{cn}{3} + 2 = cn + 2 \quad \text{stuck!}$$

$\Omega(n)$:

Assume $T(k) \ge ck$ for $k < n$

$$T\left(\frac{n}{3}\right) \ge \frac{cn}{3}$$

$$T(n) = 3T\left(\frac{n}{3}\right) + 2 \ge 3\frac{cn}{3} + 2 = cn + 2 \ge cn$$

Examine a few cases:

$$T(1) = d$$
$$T(3) = 3T(1) + 2 = 3d + 2$$
$$T(9) = 3T(3) + 2 = 3(3d + 2) + 2 = 9d + 6 + 2 = 9d + 8$$
$$T(27) = 3T(9) + 2 = 3(9d + 8) + 2 = 27d + 24 + 2 = 27d + 26$$

Hypothesis: $T(n) = nd + n - 1 = (d + 1)n - 1 \le cn - 1$

New try for $\mathrm{O}(n)$:

Assume $T(k) \le ck - 1$ for $k < n$

$$T\left(\frac{n}{3}\right) \le \frac{cn}{3} - 1$$

$$T(n) = 3T\left(\frac{n}{3}\right) + 2 \le 3\left(\frac{cn}{3} - 1\right) + 2 = cn - 3 + 2 = cn - 1$$

RECURSION TREE METHOD

Concepts

Draw tree – either for a particular *n* or for general case

Fan-out   Sub-instance size   Number of levels

Number of parents-of-leaves   Number of leaves

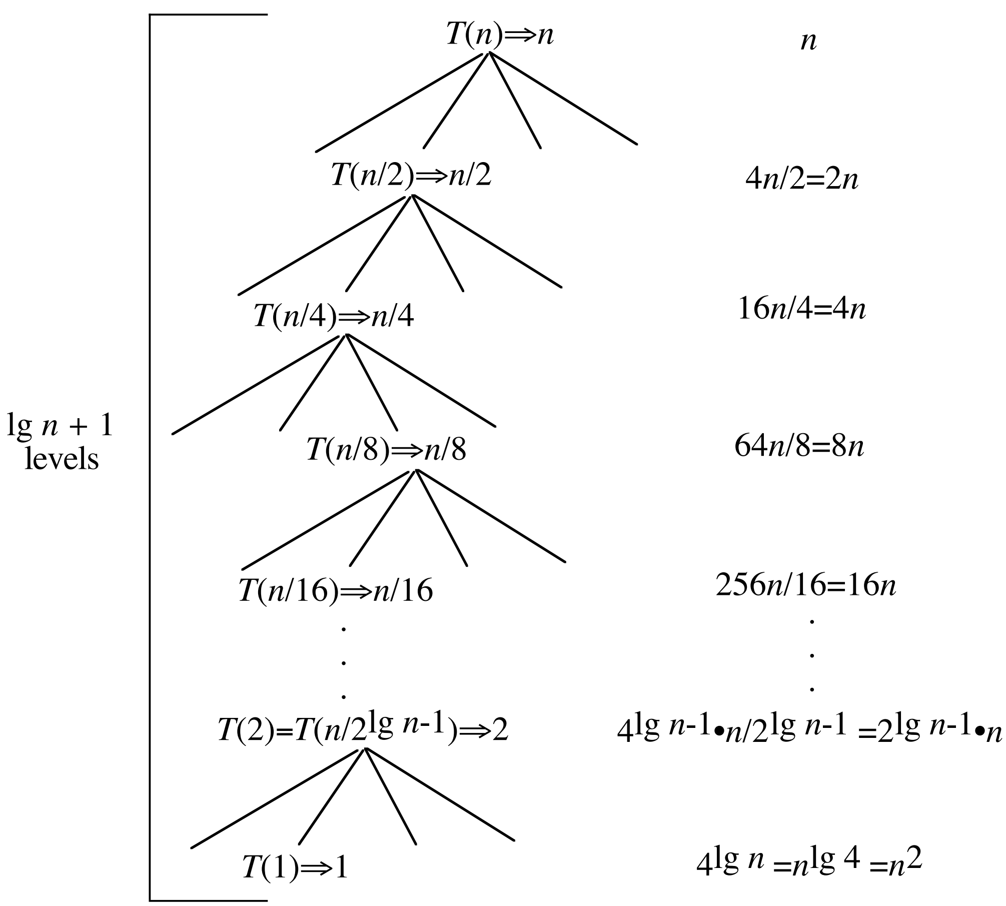
[AKA iteration: Instead of sketching tree, write an error-prone sequence of expressions.]

Assign contribution of a node for each level

Compute total contribution for each level

Usually complete by evaluating a summation – Go directly for Θ bound, not separate O and Ω.

Exercise 4.3-1a: $T(n) = 4T\left(\frac{n}{2}\right) + n$



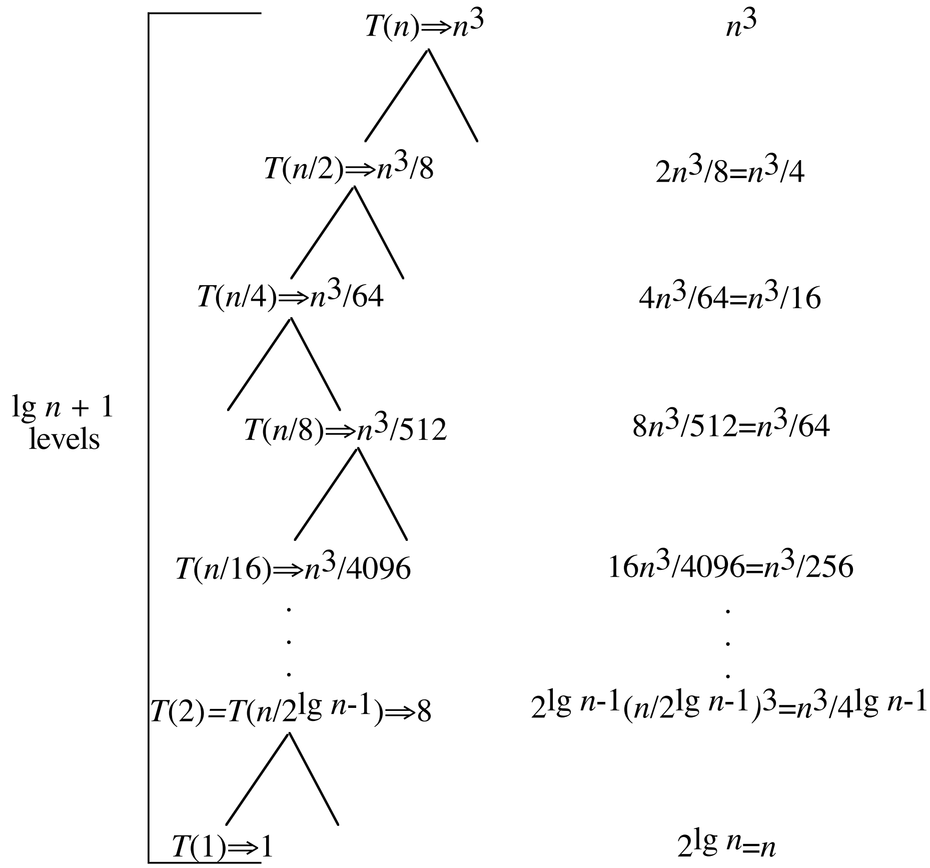
Using definite geometric sum formula:

$$n\sum_{k=0}^{\lg n-1} 2^k + n^2 = n\frac{2^{\lg n}-1}{2-1} + n^2 \qquad \text{Using } \sum_{k=0}^{t} x^k = \frac{x^{t+1}-1}{x-1} \qquad x \neq 1$$

$$= n(n-1) + n^2$$

$$= n^2 - n + n^2 = \Theta\left(n^2\right)$$

Exercise 4-1.a $\qquad T(n) = 2T\left(\frac{n}{2}\right) + n^3$



Using indefinite geometric sum formula:

$$n^3\sum_{k=0}^{\lg n-1} \frac{1}{4^k} + n \le n^3 \sum_{k=0}^{\infty} \frac{1}{4^k} + n$$

$$= n^3 \frac{1}{1-\frac{1}{4}} + n \qquad \text{From } \sum_{k=0}^{\infty} x^k = \frac{1}{1-x} \qquad 0 < x < 1$$

$$= \frac{4}{3}n^3 + n = O(n^3)$$

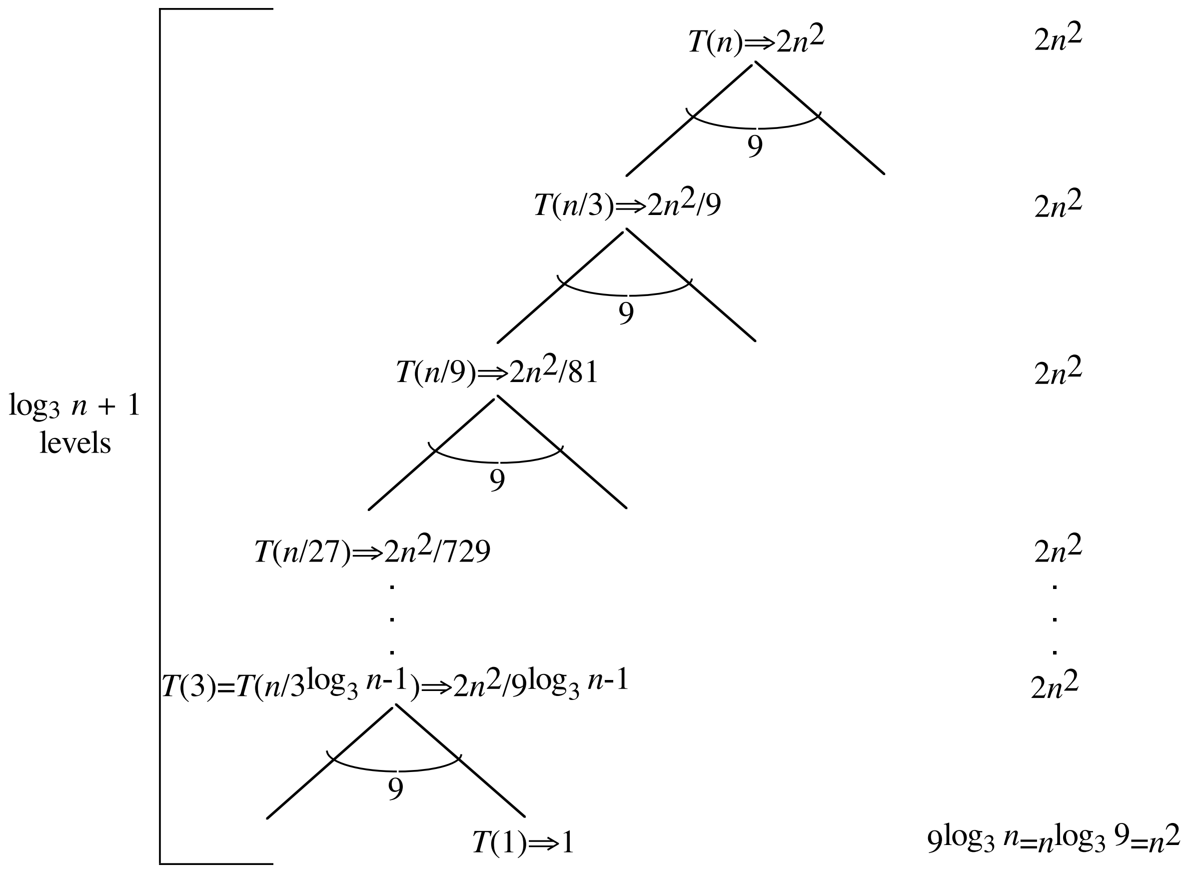
Using definite geometric sum formula:

$$n^3 \sum_{k=0}^{\lg n - 1} \frac{1}{4^k} + n = n^3 \frac{\left(\frac{1}{4}\right)^{\lg n} - 1}{\frac{1}{4} - 1} + n \qquad \text{Using } \sum_{k=0}^{t} x^k = \frac{x^{t+1} - 1}{x - 1} \qquad x \neq 1$$

$$= n^3 \frac{n^{-2} - 1}{-\frac{3}{4}} + n = n^3 \frac{1 - n^{-2}}{\frac{3}{4}} + n = \frac{4}{3} n^3 \left(1 - \frac{1}{n^2}\right) + n$$

$$= \Theta\left(n^3\right)$$

Example: $T(n) = 9T\left(\frac{n}{3}\right) + 2n^2$



$$T(n) = 2n^2 \log_3 n + n^2 = \Theta\left(n^2 \log n\right)$$