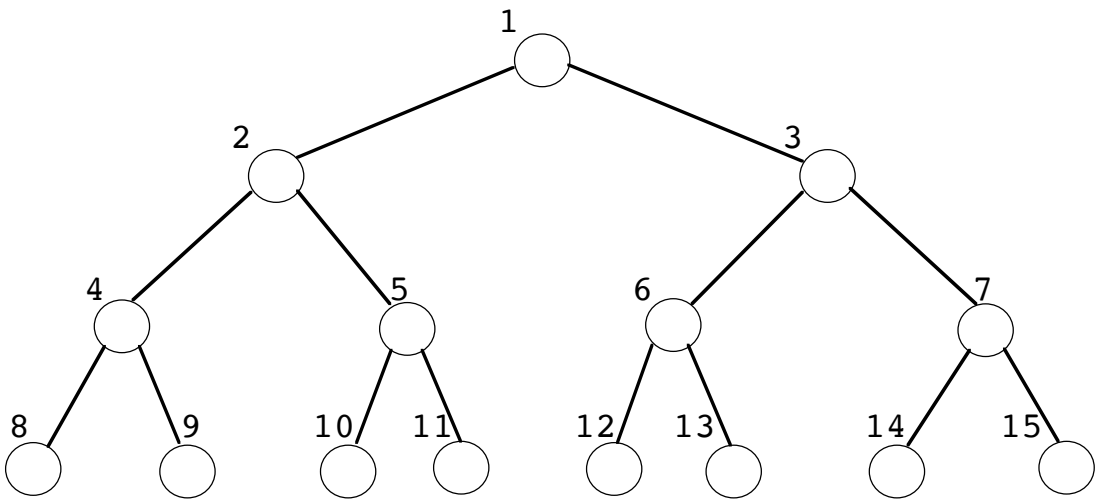
CSE 2320 Notes 5: Heapsort and Priority Queues

(Last updated 1/16/20 3:18 PM)

CLRS 6.1-6.5

5.A. (BINARY) HEAP PROPERTIES

1. Binary tree.
2. Like a complete tree, but missing some "rightmost" leaves in deepest level.

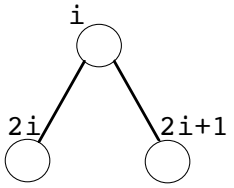


3. a. Each parent has a priority ≥ the priority of its children - maxheap.

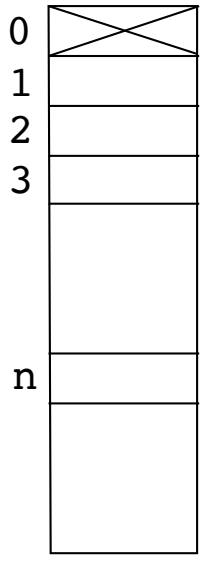
   b. Each parent has a priority ≤ the priority of its children - minheap.

Common Mapping of a Heap to an Array

1. Subscript 0 is unused.
2. Subscript 1 stores the root of the heap.
3. If it exists, the left child for subscript i is at subscript $2i$.
4. If it exists, the right child for subscript i is at subscript $2i+1$.



5. Parent of node with subscript i is $\left\lfloor \frac{i}{2} \right\rfloor$.

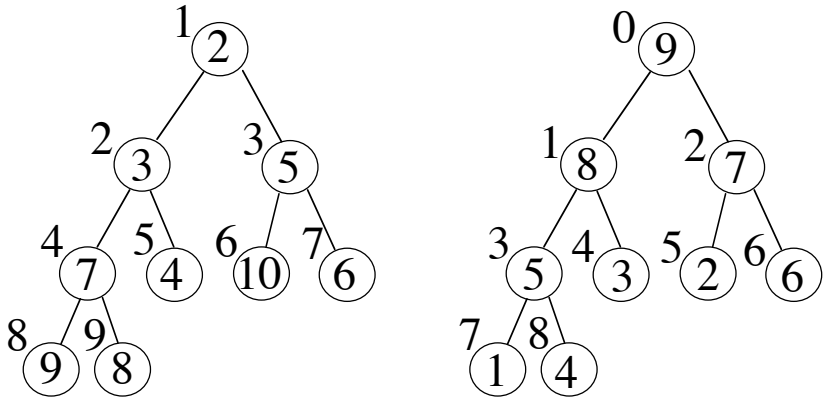


Aside: Alternate Mapping of a Heap to an Array

1. Subscript 0 stores the root of the heap.
2. If it exists, the left child for subscript i is stored at subscript $2i + 1$.
3. If it exists, the right child for subscript i is stored at subscript $2i + 2$.
4. Parent of node with subscript i is $\left\lfloor \frac{i-1}{2} \right\rfloor$.

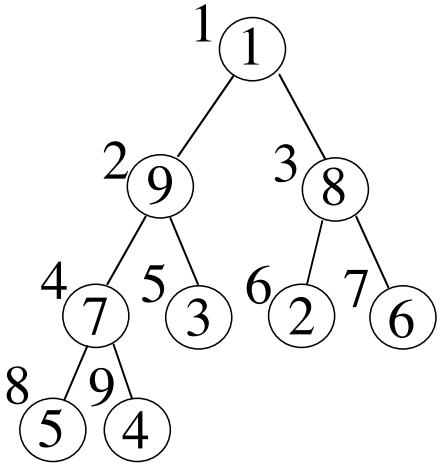
Under either mapping, the number of nodes/slots used (n) must be stored.

No pointers are needed.



5.B. CONVERTING AN UNORDERED ARRAY INTO A MAXHEAP (BOTTOM-UP HEAP CONSTRUCTION)

Special Case: Parent Node with Two Subtrees Having Maxheap Properties
(`maxHeapify/fixDown/sink`)



Worst case, $2^k \le n \le 2^{k+1} - 1$, processes _____ parents.

```
void maxHeapify(Item a[], int k, int N)
{
int j;

while (left(k) <= N)
{
  j = left(k);  // left child
  if (j < N && less(a[j], a[j+1]))
    j=right(k);    // use right child instead
  if (!less(a[k], a[j]))
    break;
  exch(a[k], a[j]); // Swap child with parent
  k = j;            // Descend
}
}
```

General Case: Move through the parents in descending subscript order, applying the special case at each parent.

```
// Convert array to maxheap, e.g. Build-Max-Heap
for (k = parent(N); k >= 1; k--)
  maxHeapify(&pq(0), k, N);
```

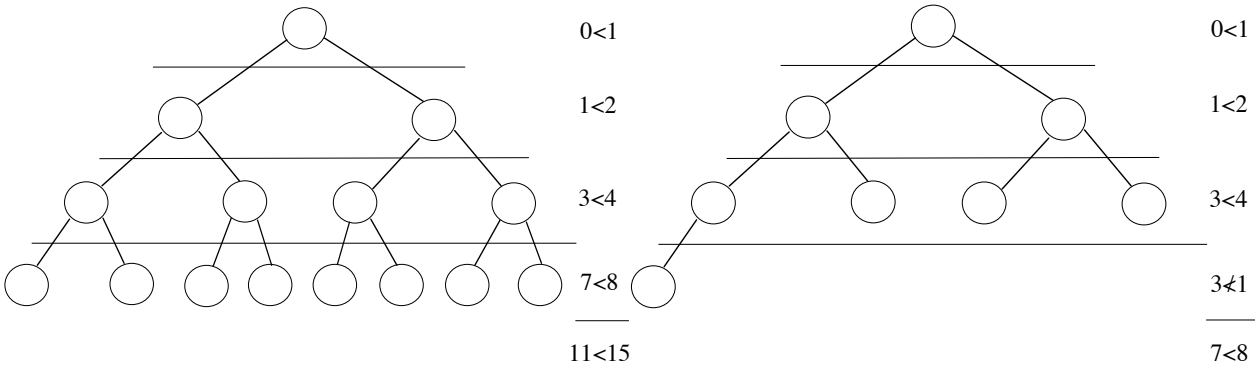
Time

Lower bound?

Worst case?

Worst case time

Each level may receive values from all ancestors in shallower levels.



5.C. SORTING USING A HEAP (`http://ranger.uta.edu/~weems/NOTES2320/heapsort.c`)

Bottom-up heap construction leaves the maximum at the root.

Swap root with element at subscript *n*. Remove subscript *n* from heap.

`maxHeapify/fixDown/sink` at root to restore maxheap property.

Time

Bottom-up heap construction takes $\Theta(n)$ for worst case.

Even though the heap loses one node after each call to `maxHeapify`, 1/2 the nodes are leaves taking $\Theta(\log n)$ worst-case time for each.

$\Theta(n \log n)$ for entire sort in worst case.

Other

Similar to selection sort, but heapsort repeatedly finds maximum instead of minimum

Heapsort is optimal, but involves larger constants than other optimal sorts

Stable?

5.D. PRIORITY QUEUES

Entry with the maximum priority is the next item to be removed.

Operations are naturally supported by a heap in $\Theta(\log n)$ worst-case time.

At least the following two operations are required to support a PQ:

`maxHeapInsert` – New item is placed at next available leaf and is swapped with ancestors having lower priorities (`fixUp/swim`).

`heapExtractMax` – Remove item with maximum priority and fix heap, just like HEAPSORT.

(http://ranger.uta.edu/~weems/NOTES2320/intPQi.c)

```
void exch(int i, int j)
{
// Swaps parent with child in heap
int t;

t = pq[i];
pq[i] = pq[j];
pq[j] = t;
qp[pq[i]] = i;
qp[pq[j]] = j;
}

int less(int i, int j)
{
// Notice how heap entries reference a[]
return a[pq[i]] < a[pq[j]];
}
```

```
void heapIncreaseKey(int *pq,int k) // AKA swim
{
while (k > 1 && less(parent(k), k))
{
  exch(k, parent(k));
  k = parent(k);
}
}

void maxHeapify(int *pq,int k, int N) // AKA sink
{
int j;

while (left(k) <= N)
{
  j = left(k);
  if (j < N && less(j, j+1))
    j=right(k);
  if (!less(k, j))
    break;
  exch(k, j);
  k = j;
}
}
```

Other convenient PQ operations, based on an "external" *dictionary*:

Dictionaries – Topic covered in detail for later exams.

Supports finding a desired (key, satellite data) pair

Some possibilities

Table (see `http://ranger.uta.edu/~weems/NOTES2320/intPQi.c` for a subscript-as-key table as dictionary)

Linked List

(Binary) Search Tree

Hash Table

Linking Between Dictionary and Heap ("Client Arrays")

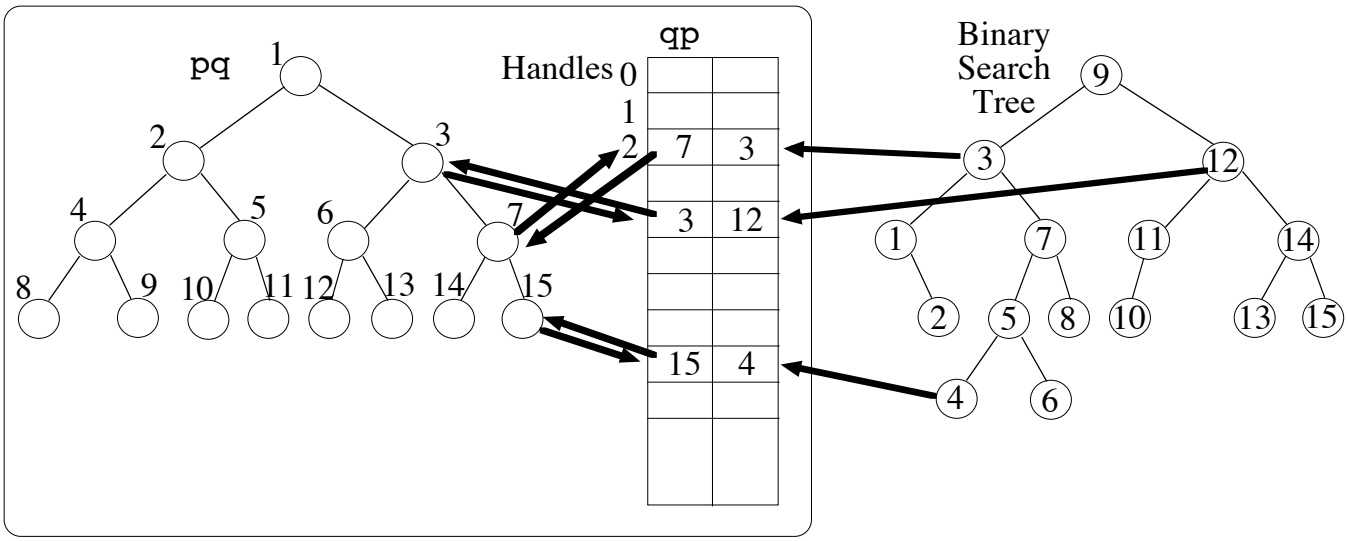
Handles (`qp`) are used to track movement of heap entries.

Satellite data in each dictionary (e.g. BST) entry includes subscript of a handle.

Priority may be stored in heap entry, handle entry, or in the dictionary (as in `intPQi.c`).

Invariants for handles that are in use: `pq[qp[i]]==i` and `qp[pq[i]]==i`

Each heap entry includes the number of its handle.



[An efficient scheme for allocating handles is considered in Notes 9, Example 3.]

Change Priority - `maxHeapChange`

`maxHeapDecreaseKey (maxHeapify/fixDown/sink)`

`maxHeapIncreaseKey (fixUp/swim)`

Delete Entry – `maxHeapDelete`

Replace with contents of highest-numbered leaf, then fix ordering.