# CSE 2320 Notes 15: Minimum Spanning Trees

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CLRS 21.3, 23.1-23.2

## 15.A. CONCEPTS

Given a weighted, connected, undirected graph, find a minimum (total) weight free tree connecting the vertices. (AKA bottleneck shortest path tree)

*Cut Property*: Suppose S and T partition V such that

1. $S \cap T = \emptyset$
2. $S \cup T = V$
3. $|S| > 0$ and $|T| > 0$

then there is some MST that includes a minimum weight edge {s, t} with $s \in S$ and $t \in T$.

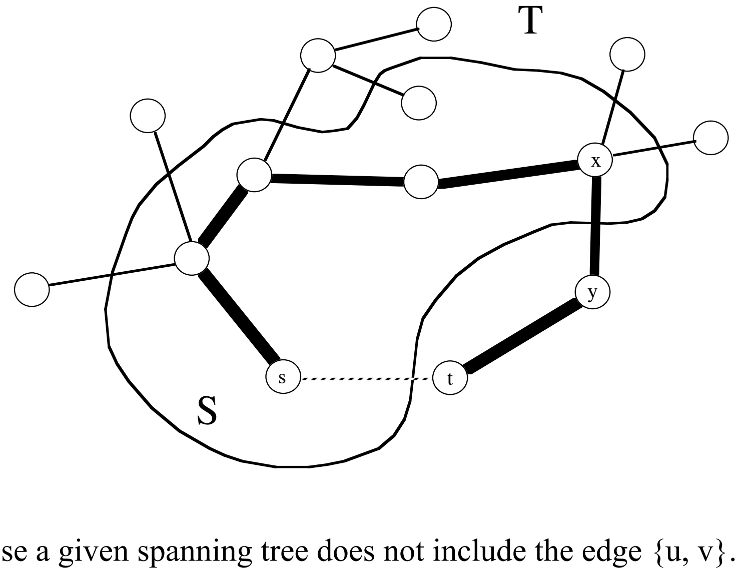
Proof:

Suppose there is a partition with a minimum weight edge {s, t}.

A spanning tree without {s, t} must still have a path between s and t.

Since $s \in S$ and $t \in T$, there must be at least one edge {x, y} on this path with $x \in S$ and $y \in T$.

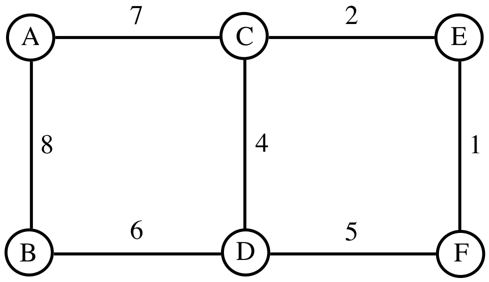
By removing {x, y} and including {s, t}, a spanning tree whose total weight is no larger is obtained. •••



*Cycle Property*: Suppose a given spanning tree does not include the edge {u, v}. If the weight of {u, v} is no larger than the weight of an edge {x, y} on the unique spanning tree path between u and v, then replacing {x, y} with {u, v} yields a spanning tree whose weight does not exceed that of the original spanning tree.

Proof: Including {u, v} in the set of chosen edges introduces a cycle, but removing {x, y} will remove the cycle to yield a modified tree whose weight is no larger.

The proof suggests a slow approach - iteratively find and remove a maximum weight edge from some remaining cycle:



## 15.B. PRIM’S ALGORITHM – Three versions

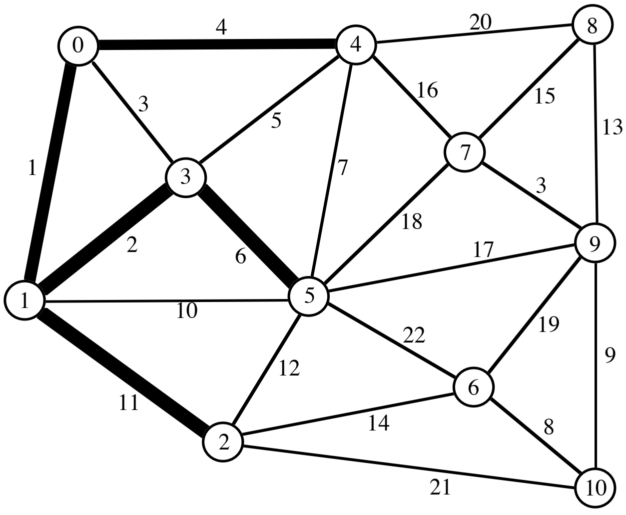
Prim’s algorithm applies the cut property by having S include those vertices connected by a subtree of the eventual MST and T contains vertices that have not yet been included. A minimum weight edge from S to T will be used to move one vertex from T to S

1. “Memoryless” – Only saves partial MST and current partition.
   ( `http://ranger.uta.edu/~weems/NOTES2320/primMemoryless.c` )

```
Place any vertex $x \in V$ in S.
$T = V - \{x\}$
while $T \neq \emptyset$
        Find the minimum weight edge $\{s, t\}$ over all $t \in T$ and all $s \in S$.  (Scan adj. list for each t)
        Include $\{s, t\}$ in MST.
        $T = T - \{t\}$
        $S = S \cup \{t\}$
```

Since no substantial data structures are used, this takes $\Theta(EV)$ time.

*Which edge does Prim’s algorithm select next?*

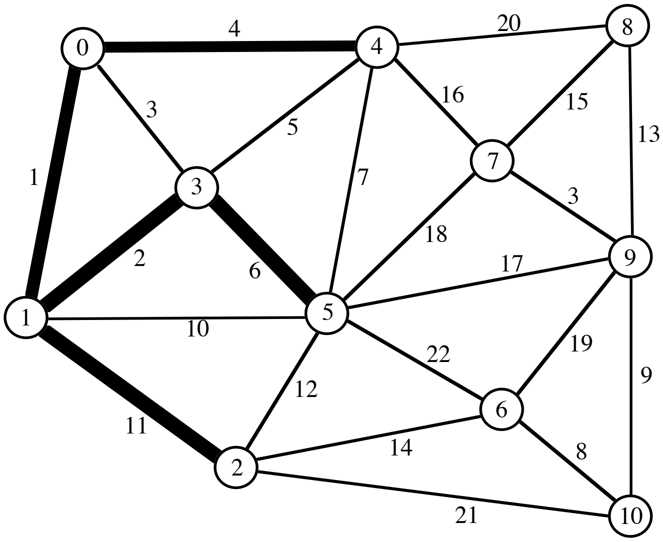


2. Maintains T-table that provides the closest vertex in S for each vertex in T.
   ( `http://ranger.uta.edu/~weems/NOTES2320/primTable.c` traverses adjacency lists)

Eliminates scanning all T adjacency lists in every phase, but still scans the adjacency list of the last vertex moved from T to S.

```
Place any vertex $x \in V$ in S.
$T = V - \{x\}$
for each $t \in T$
    Initialize T-table entry with weight of $\{t, x\}$ (or $\infty$ if non-existent) and x as best-S-neighbor
while $T \neq \emptyset$
    Scan T-table entries for the minimum weight edge {t, best-S-neighbor[t]}
        over all $t \in T$ and all $s \in S$.
    Include edge {t, best-S-neighbor[t]} in MST.
    $T = T - \{t\}$
    $S = S \cup \{t\}$
    for each vertex x in adjacency list of t
        if $x \in T$ and weight of $\{x, t\}$ < T-weight[x]
            T-weight[x] = weight of $\{x, t\}$
            best-S-neighbor[x] = t
```

*What are the T-table contents before and after the next MST vertex is selected?*



| | |
|---|---|
| 6 | 14 (2) |
| 7 | 16 (4) |
| 8 | 20 (4) |
| 9 | 17 (5) |
| 10 | 21 (2) |

Analysis:

Initializing the T-table takes $\Theta(V)$.
Scans of T-table entries contribute $\Theta(V^2)$.
Traversals of adjacency lists contribute $\Theta(E)$.
$\Theta\left(V^2 + E\right)$ overall worst-case.

3. Replace T-table by a min-heap.
   ( `http://ranger.uta.edu/~weems/NOTES2320/primHeap.cpp` )

The time for updating for best-S-neighbor increases, but the time for selection of the next vertex to move from T to S improves.

```
Place any vertex $x \in V$ in S.
$T = V - \{x\}$
for each $t \in T$
    Load T-heap entry with weight (as the priority) of $\{t, x\}$ (or $\infty$ if non-existent) and x as
        best-S-neighbor
minHeapInit(T-heap)  // a fixDown at each parent node in heap
while $T \neq \emptyset$
    Use heapExtractMin /* fixDown */ to obtain T-heap entry with the minimum weight edge
        over all $t \in T$ and all $s \in S$.
    Include edge {t, best-S-neighbor[t]} in MST.
    $T = T - \{t\}$
    $S = S \cup \{t\}$
    for each vertex x in adjacency list of t
        if $x \in T$ and weight of $\{x, t\}$ < T-weight[x]
            T-weight[x] = weight of $\{x, t\}$
            best-S-neighbor[x] = t
            minHeapChange(T-heap)  // fixUp
```

Analysis:

Initializing the T-heap takes $\Theta(V)$.
Total cost for `heapExtractMins` is $\Theta(V \log V)$.
Traversals of adjacency lists and `minHeapChange`s contribute $\Theta(E \log V)$.
$\Theta(E \log V)$ overall worst-case, since $E > V$.

*Which version is the fastest?*

| | Theory | Sparse $(E = O(V))$ | Dense $\left(E = \Omega(V^2)\right)$ |
|---|---|---|---|
| 1. | $\Theta(EV)$ | $\Theta(V^2)$ | $\Theta(V^3)$ |
| 2. | $\Theta(V^2 + E)$ | $\Theta(V^2)$ | $\Theta(V^2)$ |
| 3. | $\Theta(E \log V)$ | $\Theta(V \log V)$ | $\Theta(V^2 \log V)$ |

## 15.C. UNION-FIND TREES TO REPRESENT DISJOINT SUBSETS

Abstraction:

Set of *n* elements: 0 . . *n* - 1

Initially all elements are in *n* different subsets

`find(i)` - Returns integer ("leader") indicating which subset includes `i`

`i` and `j` are in the same subset ⇔ `find(i)==find(j)`

`union(i,j)` - Takes the set union of the subsets with leaders `i` and `j`.

Results of previous `find`s are invalid after a `union`.

Implementation 1: ( `http://ranger.uta.edu/~weems/NOTES2320/uf1.c` )

Initialization:

```
for (i=0; i<n; i++)
  id[i]=i;
```

`find(i)`:

```
return id[i];
```

`unionFunc(i,j)`:

```
for (k=0; k<n; k++)
  if (id[k]==i)
    id[k]=j;
```

| 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 |

Implementation 2: ( `http://ranger.uta.edu/~weems/NOTES2320/uf2.c` )

`find(i)`:

```
while (id[i]!=i)
  i=id[i];
return i;
```

`unionFunc(i,j)`:

```
id[i]=j;
```

| 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 |

Implementation 3: ( `http://ranger.uta.edu/~weems/NOTES2320/uf3.c` )

Initialization:

```
for (i=0; i<n; i++)
{
  id[i]=i;
  sz[i]=1;
}
```

```
find(x):

      for (i=x;
           id[i]!=i;
           i=id[i])
        ;
      root=i;
      // path compression - make all nodes on path
      // point directly at the root
      for (i=x;
           id[i]!=i;
           j=id[i],id[i]=root,i=j)
        ;
      return root;

unionFunc(i,j):

      if (sz[i]<sz[j])
      {
        id[i]=j;
        sz[j]+=sz[i];
      }
      else
      {
        id[j]=i;
        sz[i]+=sz[j];
      }
```

Best-case (shallow tree) and worst-case (deep tree) for a sequence of unions?

## 15.D. KRUSKAL'S ALGORITHM – A Simple Method for MSTs Based on Union-Find Trees

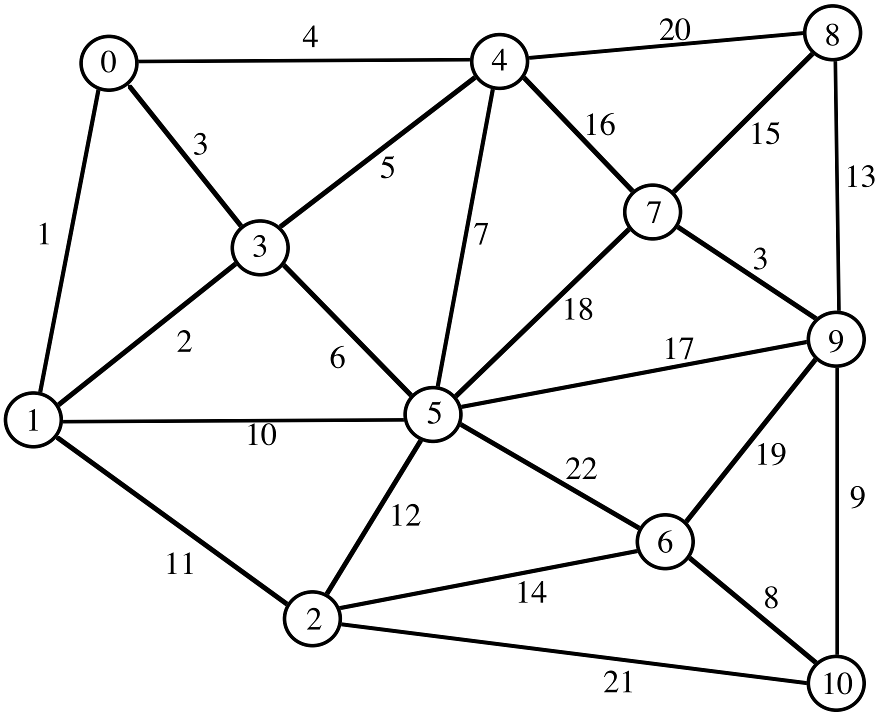
( `http://ranger.uta.edu/~weems/NOTES2320/kruskal.c` )

Sort edges in ascending weight order.

Place each vertex in its own set.

Process each edge {x, y} in sorted order:

```
    a=FIND(x)
    b=FIND(y)
    if $a \neq b$
        UNION(a,b)
        Include {x, y} in MST
```



| | |
|---|---|
| 1 | {0, 1} |
| 2 | {1, 3} |
| 3 | {0, 3} |
| 3 | {7, 9} |
| 4 | {0, 4} |
| 5 | {3, 4} |
| 6 | {3, 5} |
| 7 | {4, 5} |
| 8 | {6, 10} |
| 9 | {9, 10} |
| 10 | {1, 5} |
| 11 | {1, 2} |
| 12 | {2, 5} |
| 13 | {8, 9} |
| 14 | {2, 6} |
| ---------------- | |
| 15 | {7, 8} |
| 16 | {4, 7} |
| 17 | {5, 9} |
| 18 | {5, 7} |
| 19 | {6, 9} |
| 20 | {4, 8} |
| 21 | {2, 10} |
| 22 | {5, 6} |

Time to sort, $\Theta(E \log V)$, dominates computation