CSE 2320 Notes 16: Shortest Paths

(Last updated 11/17/18 10:21 AM)

CLRS 24.3, 25.2

16.A. CONCEPTS

(Aside: `http://dl.acm.org.ezproxy.uta.edu/citation.cfm?doid=2597757.2530531`)

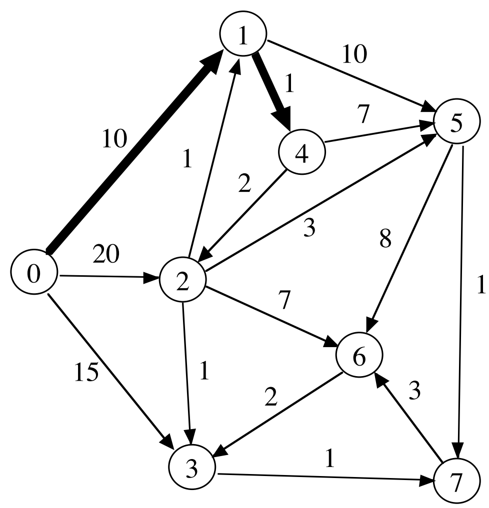
Input:

> Directed graph with *non-negative* edge weights (stored as adj. matrix for Floyd-Warshall)
> Dijkstra – source vertex

Output:

> Dijkstra – tree that gives a shortest path from source to each vertex
> Floyd-Warshall – shortest path between each pair of vertices ("all-pairs") as matrix

16.B. DIJKSTRA'S ALGORITHM – three versions



0	1	2	3	4	5	6	7
0(-)	∞	∞	∞	∞	∞	∞	∞
*	10(0)	20(0)	15(0)				
	*			11(1)	20(1)		
		13(4)		*	18(4)		

==

		*	14(2)		16(2)	20(2)	
			*				15(3)
						18(7)	*
					*		
						*	

Similar to Prim's MST:

S = vertices whose shortest path is known (initially just the source)

Length of path
Predecessor (vertex) on path (AKA shortest path tree)

T = vertices whose shortest path is not known

Each phase moves a T vertex to S by virtue of that vertex having the shortest path among all T vertices.

Third version may be viewed as being BFS with the FIFO queue replaced by a priority queue.

1. "Memoryless" – Only saves shortest path tree and current partition.
   (`http://ranger.uta.edu/~weems/NOTES2320/dijkstraMemoryless.c`)

Place desired source vertex x ∈ V in S
T = V – {x}
x.distance = 0
x.pred = (-1)
while T ≠ ∅
    Find the edge (s, t) over all t ∈ T and all s ∈ S with minimum value for s.distance + weight(s, t)
        (i.e. scan adj. list for each s)
    t.distance = s.distance + weight(s, t)
    t.pred = s
    T = T – {t}
    S = S ∪ {t}

Since no substantial data structures are used, this takes $\Theta(EV)$ time.

2. Maintains T-table that provides the predecessor vertex in S for each vertex t ∈ T to give the shortest possible path through S to t. (`http://ranger.uta.edu/~weems/NOTES2320/dijkstraTable.c`)

Eliminates scanning all S adjacency lists in every phase, but still scans the list of the last vertex moved from T to S.

Place desired source vertex x ∈ V in S
T = V – {x}
x.distance = 0
x.pred = (-1)
for each t ∈ T
    Initialize t.distance with weight of (x, t) (or ∞ if non-existent) and t.pred = x

```
while T ≠ ∅
    Scan T entries to find vertex t with minimum value for t.distance
    T = T – {t}
    S = S ∪ {t}
    for each vertex x in adjacency list of t (i.e. (t, x))
        if x ∈ T and t.distance + weight(t, x) < x.distance
            x.distance = t.distance + weight(t, x)
            x.pred = t
```

Analysis:

Initializing the T-table takes Θ(V).
Scans of T-table entries contribute $\Theta(V^2)$.
Traversals of adjacency lists contribute Θ(E).
$\Theta\left(V^2 + E\right)$ overall worst-case.

3. Replace T-table by a min-heap.
(`http://ranger.uta.edu/~weems/NOTES2320/dijkstraHeap.cpp`)

The time for updating distances and predecessors increases, but the time for selection of the next vertex to move from T to S improves.

```
Place desired source vertex x ∈ V in S
T = V – {x}
x.distance = 0
x.pred = (-1)
for each t ∈ T
    Initialize T-heap with weight (as the priority) of (x, t) (or ∞ if non-existent) and t.pred = x
minHeapInit(T-heap)  // a fixDown at each parent node in heap
while T ≠ ∅
    Use heapExtractMin /* fixDown */ to obtain T-heap entry with minimum t.distance
    T = T – {t}
    S = S ∪ {t}
    for each vertex x in adjacency list of t (i.e. (t, x))
        if x ∈ T and t.distance + weight(t, x) < x.distance
            x.distance = t.distance + weight(t, x)
            x.pred = t
            minHeapChange(T-heap)  // fixUp
```

Analysis:

Initializing the T-heap takes Θ(V).
Total cost for `heapExtractMin`s is Θ(V log V).
Traversals of adjacency lists and `minHeapChange`s contribute Θ(E log V).
$\Theta(E \log V)$ overall worst-case, since E > V.

Which version is the fastest?

	Theory	Sparse $\left(E = O(V)\right)$	Dense $\left(E = \Omega\left(V^2\right)\right)$
1.	$\Theta(EV)$	$\Theta\left(V^2\right)$	$\Theta\left(V^3\right)$
2.	$\Theta\left(V^2 + E\right)$	$\Theta\left(V^2\right)$	$\Theta\left(V^2\right)$
3.	$\Theta(E \log V)$	$\Theta(V \log V)$	$\Theta\left(V^2 \log V\right)$

16.C. FLOYD-WARSHALL ALGORITHM

Based on adjacency matrices. Will examine three versions:

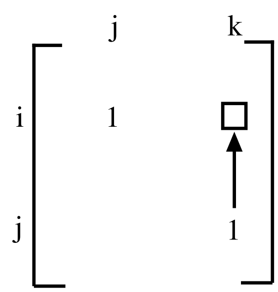
Warshall's Algorithm – After $\Theta(V^3)$ preprocessing, processes each path *existence* query in $\Theta(1)$ time.

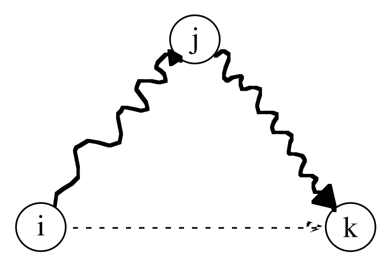
Warshall's Algorithm with Successors (or predecessors or transitive vertices) - After $\Theta(V^3)$ preprocessing, provides a path in response to a path existence query in O(V) time (similar to dynamic programming backtrace).

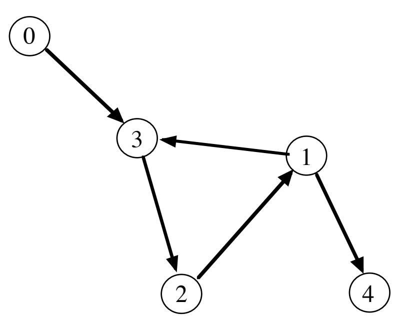
Floyd-Warshall Algorithm (with Successors) - After $\Theta(V^3)$ preprocessing, provides each *shortest* path in O(V) time.

Warshall's Algorithm:

```
for (j=0; j<V; j++)
  for (i=0; i<V; i++)
    if (A[i][j])
      for (k=0; k<V; k++)
        if (A[j][k])
          A[i][k]=1;
```







	0	1	2	3	4
0				1	
1				1	1
2		1			
3			1		
4					

If zero-edge paths are useful for an application (i.e. reflexive, self-loops), the diagonal may be all ones.

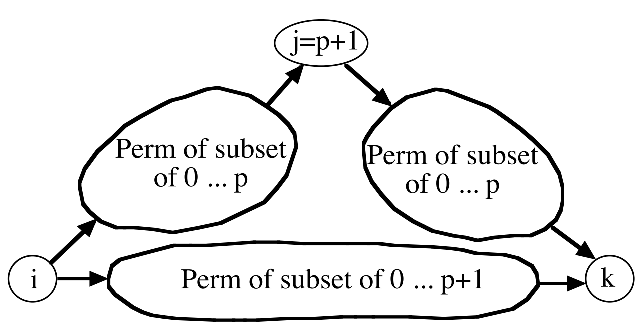
Why does it work?

a. *Correct* in use of transitivity.

b. Is it *complete*?

When	Paths That Can Be Detected
Before j=0	x → y
After j=0	x → 0 → y
After j=1	x → 1 → y x → 0 → 1 → y x → 1 → 0 → y
After j=2	x → 2 → y x → 0 → 2 → y x → 1 → 2 → y x → 2 → 0 → y x → 2 → 1 → y x → 0 → 1 → 2 → y x → 0 → 2 → 1 → y x → 1 → 0 → 2 → y x → 1 → 2 → 0 → y x → 2 → 0 → 1 → y x → 2 → 1 → 0 → y
. . .	
After j=p	x → Permutation of *subset* of 0 … p → y
After j=V-1	ALL PATHS

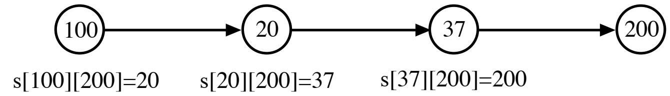
Math. Induction:



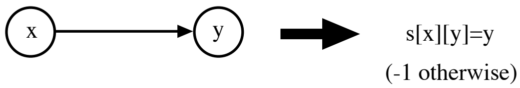
Warshall's Algorithm with Successors

Successor Matrix

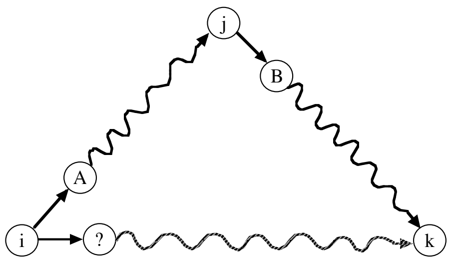
Buc-ee's directions:



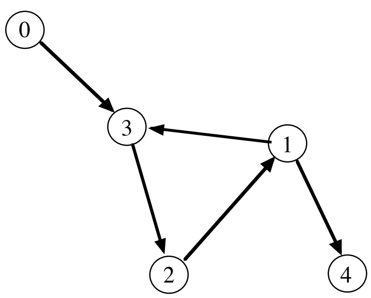
Initialize:



Warshall Matrix Update:



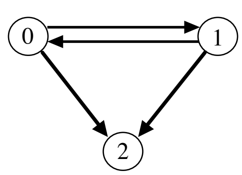
succ[i][j] = A   succ[j][k] = B   succ[i][k] = ?



	0	1	2	3	4
0				3	
1				3	4
2		1			
3			2		
4					

```
for (j=0; j<V; j++)
  for (i=0; i<V; i++)
    if (s[i][j] != (-1))
      for (k=0; k<V; k++)
        if (succ[i][k]==(-1) && succ[j][k]!=(-1))
          succ[i][k] = succ[i][j];
```

Suppose code in box is removed for this graph:



Complete Example (`http://ranger.uta.edu/~weems/NOTES2320/warshall.c`) saving paths using successors:

	0	1	2	3	4
0	-1	-1	-1	3	-1
1	-1	-1	-1	3	4
2	-1	1	-1	-1	-1
3	-1	-1	2	-1	-1
4	-1	-1	-1	-1	-1

	0	1	2	3	4
0	-1	-1	-1	3	-1
1	-1	-1	-1	3	4
2	-1	1	-1	-1	-1
3	-1	-1	2	-1	-1
4	-1	-1	-1	-1	-1

	0	1	2	3	4
0	-1	-1	-1	3	-1
1	-1	-1	-1	3	4
2	-1	1	-1	1	1
3	-1	-1	2	-1	-1
4	-1	-1	-1	-1	-1

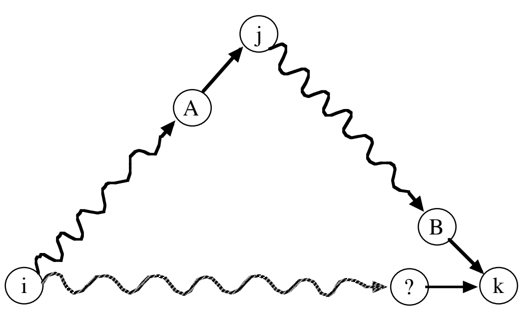
	0	1	2	3	4
0	-1	-1	-1	3	-1
1	-1	-1	-1	3	4
2	-1	1	-1	1	1
3	-1	2	2	2	2
4	-1	-1	-1	-1	-1

	0	1	2	3	4
0	-1	3	3	3	3
1	-1	3	3	3	4
2	-1	1	1	1	1
3	-1	2	2	2	2
4	-1	-1	-1	-1	-1

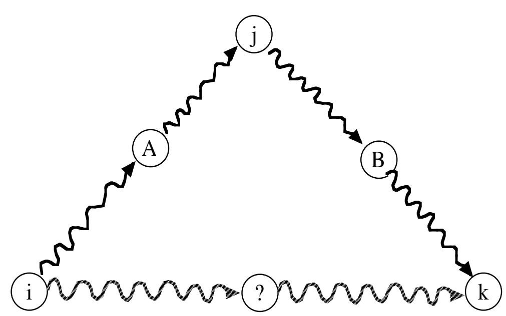
	0	1	2	3	4
0	-1	3	3	3	3
1	-1	3	3	3	4
2	-1	1	1	1	1
3	-1	2	2	2	2
4	-1	-1	-1	-1	-1

Other ways to save path information:

Predecessors (`warshallPred.c`)



Transitive/Intermediate/Column (`warshallCol.c`)

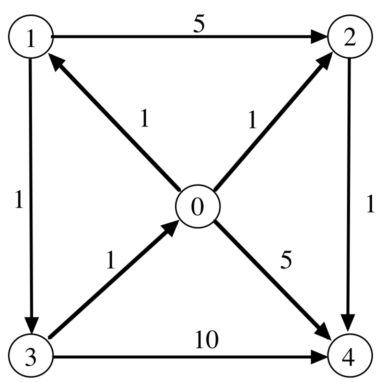



```
for (j=0;j<n;j++)
  for (i=0;i<n;i++)
    if (pred[i][j]!=(-1))
      for (k=0;k<n;k++)
        if (pred[i][k]==(-1) && pred[j][k]!=(-1))
          pred[i][k]=pred[j][k];
```

Floyd-Warshall Algorithm with Successors (`http://ranger.uta.edu/~weems/NOTES2320/floydWarshall.c`)

After j = p has been processed, the *shortest path* from each x to each y that uses *only* vertices in 0 . . . p as intermediate vertices is recorded in matrix.

```
for (j=0;j<n;j++)
{
  for (i=0;i<n;i++)
    if (dist[i][j]<oo)
      for (k=0;k<n;k++)
        if (dist[j][k]<oo)
        {
          newDist=dist[i][j]+dist[j][k];
          if (newDist<dist[i][k])
          {
            dist[i][k]=newDist;
            succ[i][k]=succ[i][j];
          }
        }
}
```



	0	1	2	3	4
0		1	1		5
1			5	1	
2					1
3	1				10
4					

	0	1	2	3	4
0	oo	1 1	1 2	oo	5 4
1	oo	oo	5 2	1 3	oo
2	oo	oo	oo	oo	1 4
3	1 0	oo	oo	oo	10 4
4	oo	oo	oo	oo	oo

	0	1	2	3	4
0	oo	1 1	1 2	oo	5 4
1	oo	oo	5 2	1 3	oo
2	oo	oo	oo	oo	1 4
3	1 0	2 0	2 0	oo	6 0
4	oo	oo	oo	oo	oo

	0	1	2	3	4
0	oo	1 1	1 2	2 1	5 4
1	oo	oo	5 2	1 3	oo
2	oo	oo	oo	oo	1 4
3	1 0	2 0	2 0	3 0	6 0
4	oo	oo	oo	oo	oo

	0	1	2	3	4
0	3 1	1 1	1 2	2 1	2 2
1	2 3	3 3	3 3	1 3	4 3
2	oo	oo	oo	oo	1 4
3	1 0	2 0	2 0	3 0	3 0
4	oo	oo	oo	oo	oo

	0	1	2	3	4
0	oo	1 1	1 2	2 1	2 2
1	oo	oo	5 2	1 3	6 2
2	oo	oo	oo	oo	1 4
3	1 0	2 0	2 0	3 0	3 0
4	oo	oo	oo	oo	oo

	0	1	2	3	4
0	3 1	1 1	1 2	2 1	2 2
1	2 3	3 3	3 3	1 3	4 3
2	oo	oo	oo	oo	1 4
3	1 0	2 0	2 0	3 0	3 0
4	oo	oo	oo	oo	oo

Note: In this example, zero-edge paths are not considered.