CSE 5311 Notes 0: Review of Dynamic Programming

(Last updated 8/14/16 1:03 PM, extracted from CSE 2320 Notes 7)

DYNAMIC PROGRAMMING APPROACH

1. Describe problem input.

2. Determine cost function and base case.

3. Determine general case for cost function. THE HARD PART!!!

4. Appropriate ordering for enumerating subproblems.

   a. Simple bottom-up approach - from small problems towards the entire big problem.

   b. Top-down approach with "memoization" - to attack large problems.

5. Backtrace for solution. *Most of the effort in dynamic programming is ignored at the end.*

   a. Predecessor/back pointers to get to the subproblems whose results are in the solution.

   b. Top-down recomputation of cost function (to reach the same subproblems as 5.a)

   (Providing all solutions is an extra cost feature . . .)

7.B. WEIGHTED INTERVAL SCHEDULING

Input: A set of n intervals numbered 1 through n with each interval i having start time s_i, finish time f_i, and positive weight v_i,

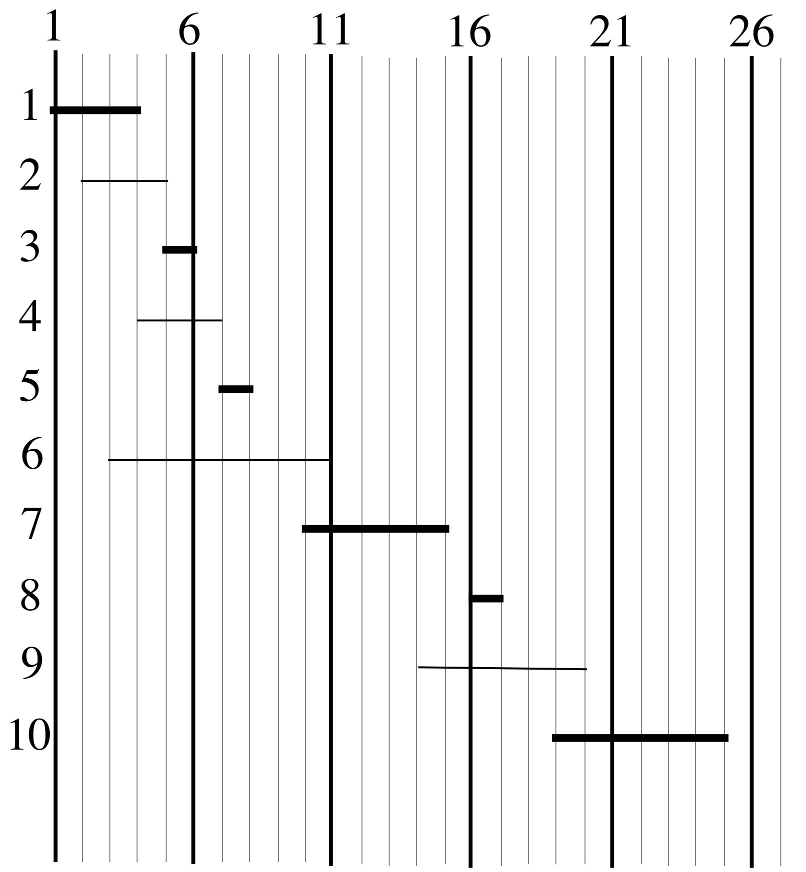
Output: A set of non-overlapping intervals to ***maximize*** the sum of their weights. (Two intervals i and j overlap if either $s_i < s_j < f_i$ or $s_i < f_j < f_i$.)

Brute-force solution: Enumerate the powerset of the input intervals, discard those cases with overlapping intervals, and compute the sum of the weights for each.

1. Describe problem input.

   Assume the n intervals are in ascending finish time order, i.e. $f_i \le f_{i+1}$.

   Let p_i be the *rightmost preceding interval* for interval i, i.e. the largest value $j < i$ such that intervals i and j do not overlap. If no such interval j exists, $p_i = 0$. (These values may be computed in $\Theta(n \log n)$ time using `binSearchLast()` from Notes 1.
   `http://ranger.uta.edu/~weems/NOTES2320/wis.bs.c`)



i	v_i	p_i	$m(i)$
1	5	0	
2	4	0	
3	3	2	
4	2	1	
5	1	4	
6	5	0	
7	4	5	
8	3	7	
9	2	6	
10	1	8	

2. Determine cost function and base case.

   $M(i)$ = Cost for optimal non-overlapping subset for the first *i* input intervals.

   $M(0) = 0$

3. Determine general case.

   For $M(i)$, the main issue is: *Does the optimal subset include interval i?*

   If *yes*: optimal subset cannot include any overlapping intervals, so $M(i) = M(p_i) + v_i$.

   If *no*: optimal subset is the same as for $M(i-1)$, so $M(i) = M(i-1)$.

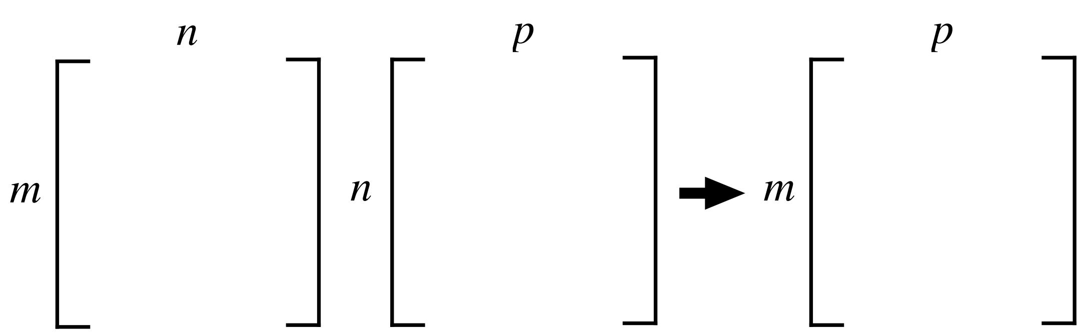
   This observation tells us to compute cost ***both*** ways and keep the maximum.

4. Appropriate ordering of subproblems. Simply compute $M(i)$ in ascending *i* order.

5. Backtrace for solution (with recomputation). This is the subset of intervals for $M(n)$.

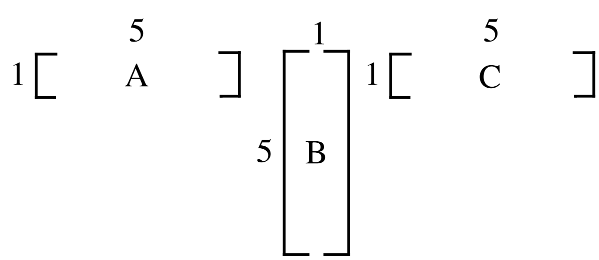
```
i=n;
while (i>0)
  if (v[i]+M[p[i]]>=M[i-1])
  {
    // Interval i is in solution
    i=p[i];
  }
  else
    i--;
```

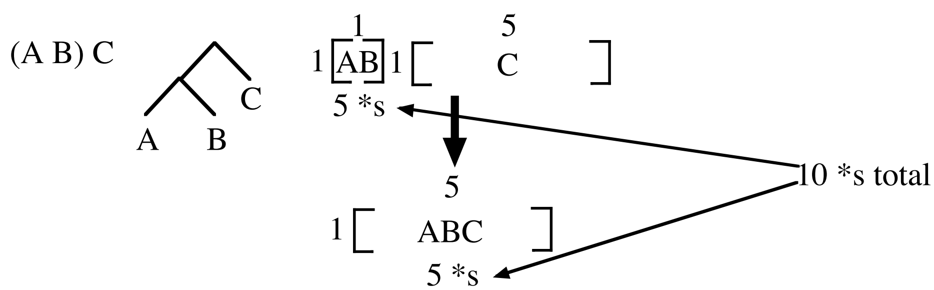
7.C. OPTIMAL MATRIX MULTIPLICATION ORDERING (very simplified version of query optimization)

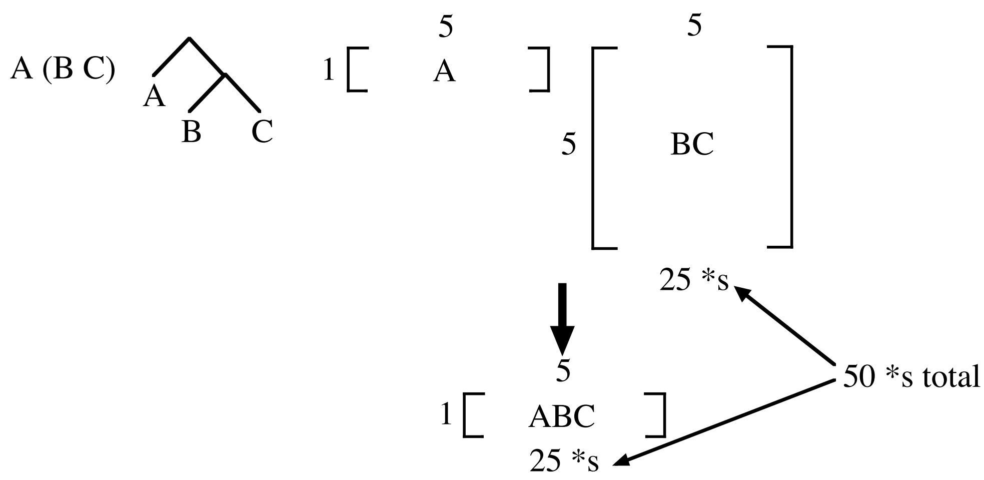


Only one strategy for multiplying two matrices – requires *mnp* scalar multiplications (and $m(n - 1)p$ additions).

There are two strategies for multiplying three matrices:







Aside: Ways to parenthesize n matrices? (Catalan numbers)

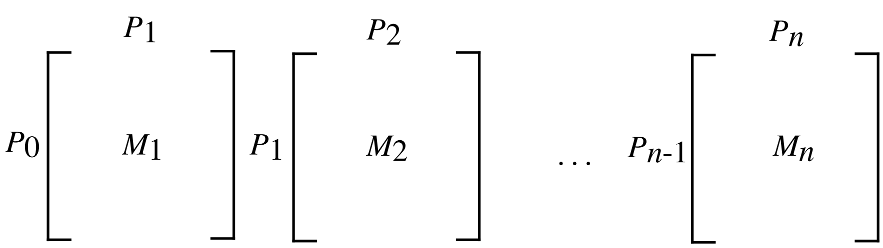
$$C_0 = 1 \qquad C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i} \text{ for } n \geq 0 \qquad C_n = \frac{1}{n+1}\binom{2n}{n}$$

(`http://en.wikipedia.org/wiki/Catalan_number`)

Observation: Final tree cannot be optimal if any subtree is not.

1. Describe problem input.

   n matrices $\Rightarrow$ $n + 1$ sizes



2. Determine cost function and base case.

   $C(i, j)$ = Cost for optimally multiplying $M_i \, . \, . \, . \, M_j$

   $C(i, i) = 0$

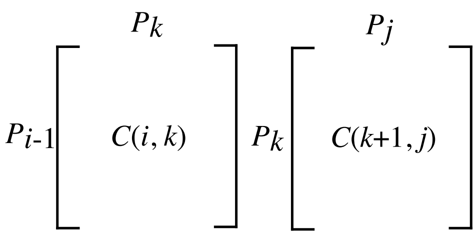
3. Determine general case.

   Consider a specific case $C(5, 9)$. The optimal way to multiply $M_5 \ldots M_9$ could be any of the following:

   $C(5, 5) + C(6, 9) + P_4 P_5 P_9$
   $C(5, 6) + C(7, 9) + P_4 P_6 P_9$
   $C(5, 7) + C(8, 9) + P_4 P_7 P_9$
   $C(5, 8) + C(9, 9) + P_4 P_8 P_9$

   Compute all four and keep the smallest one.

   Abstractly: Trying to find $C(i, j)$



$$C(i, j) = \min_{i \le k < j} \left\{ C(i,k) + C(k+1, j) + P_{i-1} P_k P_j \right\}$$

4. Appropriate ordering of subproblems.

   Since smaller subproblems are needed to solve larger problems, run value for $j - i$ for $C(i, j)$ from 0 to $n - 1$. Suppose $n = 5$:

0	1	2	3	4
$C(1,1)$	$C(1,2)$	$C(1,3)$	$C(1,4)$	$C(1,5)$
$C(2,2)$	$C(2,3)$	$C(2,4)$	$C(2,5)$	
$C(3,3)$	$C(3,4)$	$C(3,5)$		
$C(4,4)$	$C(4,5)$			
$C(5,5)$				

5. Backtrace for solution – explicitly save the k value that gave each $C(i, j)$.

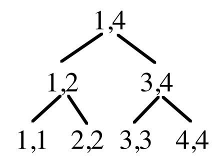
http://ranger.uta.edu/~weems/NOTES2320/optMM.c

```
// Optimal matrix multiplication order using dynamic programming
#include <stdio.h>
main()
{
int p[20];
int n;
int c[20][20];
int trace[20][20];

int i,j,k;
int work;

scanf("%d",&n);
for (i=0;i<=n;i++)
  scanf("%d",&p[i]);
for (i=1;i<=n;i++)
  c[i][i]=trace[i][i]=0;
for (i=1;i<n;i++)
  for (j=1;j<=n-i;j++)
  {
    printf("Compute c[%d][%d]\n",j,j+i);
    c[j][j+i]=999999;
    for (k=j;k<j+i;k++)
    {
      work=c[j][k]+c[k+1][j+i]+p[j-1]*p[k]*p[j+i];
      printf("  k=%d gives cost %3d=c[%d][%d]+c[%d][%d]+p[%d]*p[%d]*p[%d]\n",
        k,work,j,k,k+1,j+i,j-1,k,j+i);
      if (c[j][j+i]>work)
      {
        c[j][j+i]=work;
        trace[j][j+i]=k;
      }
    }
    printf("  c[%d][%d]==%d,trace[%d][%d]==%d\n",j,j+i,
      c[j][j+i],j,j+i,trace[j][j+i]);
  }

printf("   ");
for (i=1;i<=n;i++)
  printf("   %3d   ",i);
printf("\n");
for (i=1;i<=n;i++)
{
  printf("%2d ",i);
  for (j=1;j<=n;j++)
    if (i>j)
      printf(" ------- ");
    else
      printf(" %3d %3d ",c[i][j],trace[i][j]);
  printf("\n");
  printf("\n");
}
}
```

It is straightforward to use integration to determine that the k loop body executes about $\frac{n^3}{6}$ times.

```
4
2 4 3 5 2
Compute c[1][2]
  k=1 gives cost  24=c[1][1]+c[2][2]+p[0]*p[1]*p[2]
  c[1][2]==24,trace[1][2]==1
Compute c[2][3]
  k=2 gives cost  60=c[2][2]+c[3][3]+p[1]*p[2]*p[3]
  c[2][3]==60,trace[2][3]==2
Compute c[3][4]
  k=3 gives cost  30=c[3][3]+c[4][4]+p[2]*p[3]*p[4]
  c[3][4]==30,trace[3][4]==3
Compute c[1][3]
  k=1 gives cost 100=c[1][1]+c[2][3]+p[0]*p[1]*p[3]
  k=2 gives cost  54=c[1][2]+c[3][3]+p[0]*p[2]*p[3]
  c[1][3]==54,trace[1][3]==2
Compute c[2][4]
  k=2 gives cost  54=c[2][2]+c[3][4]+p[1]*p[2]*p[4]
  k=3 gives cost 100=c[2][3]+c[4][4]+p[1]*p[3]*p[4]
  c[2][4]==54,trace[2][4]==2
```

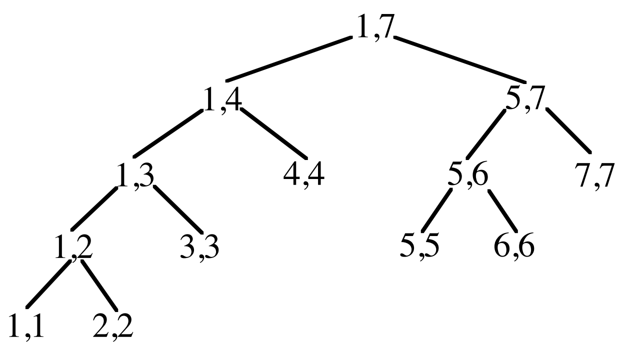
```
Compute c[1][4]
  k=1 gives cost  70=c[1][1]+c[2][4]+p[0]*p[1]*p[4]
  k=2 gives cost  66=c[1][2]+c[3][4]+p[0]*p[2]*p[4]
  k=3 gives cost  74=c[1][3]+c[4][4]+p[0]*p[3]*p[4]
  c[1][4]==66,trace[1][4]==2
         1           2           3           4
 1    0    0    24    1    54    2    66    2

 2  -------     0    0    60    2    54    2

 3  -------  -------     0    0    30    3

 4  -------  -------  -------     0    0
```




```
7
1 7 9 5 1 5 10 3
Compute c[1][2]
  k=1 gives cost  63=c[1][1]+c[2][2]+p[0]*p[1]*p[2]
  c[1][2]==63,trace[1][2]==1
Compute c[2][3]
  k=2 gives cost 315=c[2][2]+c[3][3]+p[1]*p[2]*p[3]
  c[2][3]==315,trace[2][3]==2
Compute c[3][4]
  k=3 gives cost  45=c[3][3]+c[4][4]+p[2]*p[3]*p[4]
  c[3][4]==45,trace[3][4]==3
Compute c[4][5]
  k=4 gives cost  25=c[4][4]+c[5][5]+p[3]*p[4]*p[5]
  c[4][5]==25,trace[4][5]==4
Compute c[5][6]
  k=5 gives cost  50=c[5][5]+c[6][6]+p[4]*p[5]*p[6]
  c[5][6]==50,trace[5][6]==5
Compute c[6][7]
  k=6 gives cost 150=c[6][6]+c[7][7]+p[5]*p[6]*p[7]
  c[6][7]==150,trace[6][7]==6
Compute c[1][3]
  k=1 gives cost 350=c[1][1]+c[2][3]+p[0]*p[1]*p[3]
  k=2 gives cost 108=c[1][2]+c[3][3]+p[0]*p[2]*p[3]
  c[1][3]==108,trace[1][3]==2
Compute c[2][4]
  k=2 gives cost 108=c[2][2]+c[3][4]+p[1]*p[2]*p[4]
  k=3 gives cost 350=c[2][3]+c[4][4]+p[1]*p[3]*p[4]
  c[2][4]==108,trace[2][4]==2
Compute c[3][5]
  k=3 gives cost 250=c[3][3]+c[4][5]+p[2]*p[3]*p[5]
  k=4 gives cost  90=c[3][4]+c[5][5]+p[2]*p[4]*p[5]
  c[3][5]==90,trace[3][5]==4
Compute c[4][6]
  k=4 gives cost 100=c[4][4]+c[5][6]+p[3]*p[4]*p[6]
  k=5 gives cost 275=c[4][5]+c[6][6]+p[3]*p[5]*p[6]
  c[4][6]==100,trace[4][6]==4
Compute c[5][7]
  k=5 gives cost 165=c[5][5]+c[6][7]+p[4]*p[5]*p[7]
  k=6 gives cost  80=c[5][6]+c[7][7]+p[4]*p[6]*p[7]
  c[5][7]==80,trace[5][7]==6
Compute c[1][4]
  k=1 gives cost 115=c[1][1]+c[2][4]+p[0]*p[1]*p[4]
  k=2 gives cost 117=c[1][2]+c[3][4]+p[0]*p[2]*p[4]
  k=3 gives cost 113=c[1][3]+c[4][4]+p[0]*p[3]*p[4]
  c[1][4]==113,trace[1][4]==3
Compute c[2][5]
  k=2 gives cost 405=c[2][2]+c[3][5]+p[1]*p[2]*p[5]
  k=3 gives cost 515=c[2][3]+c[4][5]+p[1]*p[3]*p[5]
  k=4 gives cost 143=c[2][4]+c[5][5]+p[1]*p[4]*p[5]
  c[2][5]==143,trace[2][5]==4
```

```
Compute c[3][6]
  k=3 gives cost 550=c[3][3]+c[4][6]+p[2]*p[3]*p[6]
  k=4 gives cost 185=c[3][4]+c[5][6]+p[2]*p[4]*p[6]
  k=5 gives cost 540=c[3][5]+c[6][6]+p[2]*p[5]*p[6]
  c[3][6]==185,trace[3][6]==4
Compute c[4][7]
  k=4 gives cost  95=c[4][4]+c[5][7]+p[3]*p[4]*p[7]
  k=5 gives cost 250=c[4][5]+c[6][7]+p[3]*p[5]*p[7]
  k=6 gives cost 250=c[4][6]+c[7][7]+p[3]*p[6]*p[7]
  c[4][7]==95,trace[4][7]==4
Compute c[1][5]
  k=1 gives cost 178=c[1][1]+c[2][5]+p[0]*p[1]*p[5]
  k=2 gives cost 198=c[1][2]+c[3][5]+p[0]*p[2]*p[5]
  k=3 gives cost 158=c[1][3]+c[4][5]+p[0]*p[3]*p[5]
  k=4 gives cost 118=c[1][4]+c[5][5]+p[0]*p[4]*p[5]
  c[1][5]==118,trace[1][5]==4
Compute c[2][6]
  k=2 gives cost 815=c[2][2]+c[3][6]+p[1]*p[2]*p[6]
  k=3 gives cost 765=c[2][3]+c[4][6]+p[1]*p[3]*p[6]
  k=4 gives cost 228=c[2][4]+c[5][6]+p[1]*p[4]*p[6]
  k=5 gives cost 493=c[2][5]+c[6][6]+p[1]*p[5]*p[6]
  c[2][6]==228,trace[2][6]==4
Compute c[3][7]
  k=3 gives cost 230=c[3][3]+c[4][7]+p[2]*p[3]*p[7]
  k=4 gives cost 152=c[3][4]+c[5][7]+p[2]*p[4]*p[7]
  k=5 gives cost 375=c[3][5]+c[6][7]+p[2]*p[5]*p[7]
  k=6 gives cost 455=c[3][6]+c[7][7]+p[2]*p[6]*p[7]
  c[3][7]==152,trace[3][7]==4
Compute c[1][6]
  k=1 gives cost 298=c[1][1]+c[2][6]+p[0]*p[1]*p[6]
  k=2 gives cost 338=c[1][2]+c[3][6]+p[0]*p[2]*p[6]
  k=3 gives cost 258=c[1][3]+c[4][6]+p[0]*p[3]*p[6]
  k=4 gives cost 173=c[1][4]+c[5][6]+p[0]*p[4]*p[6]
  k=5 gives cost 168=c[1][5]+c[6][6]+p[0]*p[5]*p[6]
  c[1][6]==168,trace[1][6]==5
Compute c[2][7]
  k=2 gives cost 341=c[2][2]+c[3][7]+p[1]*p[2]*p[7]
  k=3 gives cost 515=c[2][3]+c[4][7]+p[1]*p[3]*p[7]
  k=4 gives cost 209=c[2][4]+c[5][7]+p[1]*p[4]*p[7]
  k=5 gives cost 398=c[2][5]+c[6][7]+p[1]*p[5]*p[7]
  k=6 gives cost 438=c[2][6]+c[7][7]+p[1]*p[6]*p[7]
  c[2][7]==209,trace[2][7]==4
Compute c[1][7]
  k=1 gives cost 230=c[1][1]+c[2][7]+p[0]*p[1]*p[7]
  k=2 gives cost 242=c[1][2]+c[3][7]+p[0]*p[2]*p[7]
  k=3 gives cost 218=c[1][3]+c[4][7]+p[0]*p[3]*p[7]
  k=4 gives cost 196=c[1][4]+c[5][7]+p[0]*p[4]*p[7]
  k=5 gives cost 283=c[1][5]+c[6][7]+p[0]*p[5]*p[7]
  k=6 gives cost 198=c[1][6]+c[7][7]+p[0]*p[6]*p[7]
  c[1][7]==196,trace[1][7]==4
```

	1		2		3		4		5		6		7	
1	0	0	63	1	108	2	113	3	118	4	168	5	196	4
2	-------		0	0	315	2	108	2	143	4	228	4	209	4
3	-------		-------		0	0	45	3	90	4	185	4	152	4
4	-------		-------		-------		0	0	25	4	100	4	95	4
5	-------		-------		-------		-------		0	0	50	5	80	6
6	-------		-------		-------		-------		-------		0	0	150	6
7	-------		-------		-------		-------		-------		-------		0	0



7.E. LONGEST INCREASING SUBSEQUENCE

Monotone: For an input sequence $Y = y_1 y_2 \ldots y_n$, find a longest subsequence in increasing (≤) order.

Strict: For an input sequence $Y = y_1 y_2 \ldots y_n$, find a longest subsequence in *strictly* increasing (<) order.

Both versions may be solved in $\Theta(n \log n)$ worst-case time, using an appropriate DP cost function and *n* binary searches.

Monotone (`http://ranger.uta.edu/~weems/NOTES2320/LIS.c`):

1. Describe problem input. $Y = y_1 y_2 \ldots y_n$

2. Determine cost function and base case.

   $C(i) =$ Length of longest increasing subsequence ending with y_i.

   $C(0) = 0$

3. Determine general case for cost function.

   $C(i) = 1 + \max_{j<i \text{ and } y_j \le y_i} \{C(j)\}$ (The j that gives $C(i)$ may be saved for backtrace.)

4. Appropriate ordering of subproblems - iterate over the prefix length, saving C and j for each i.

i	1	2	3	4	5	6	7	8	9	10
y_i	60	10	70	80	20	10	30	40	70	20
C	1	1	2	3	2	2	3	4	5	3
j	0	0	2	3	2	2	6	7	8	6

5. Backtrace for solution.

Find the rightmost occurence of the maximum C value. The corresponding y will be minimized.

Appears to take $\Theta(n^2)$, but `binSearchLast()` from Notes 1 may be used to find each C and j pair in $\Theta(\log n)$ time to give $\Theta(n \log n)$ overall:

```
// Initialize table for binary search for DP
bsTabC[0]=(-999999);  // Must be smaller than all input values.
bsTabI[0]=0;          // Index of predecessor (0=grounded)
for (i=1;i<=n;i++)
  bsTabC[i]=999999;   // Must be larger than all input values.

C[0]=0;  // DP base case
j[0]=0;

for (i=1;i<=n;i++)
{
  // Find IS that y[i] could be appended to.
  // See CSE 2320 Notes 01 for binSearchLast()
  k=binSearchLast(bsTabC,n+1,y[i]);
  C[i]=k+1;          // Save length of LIS for y[i]
  j[i]=bsTabI[k];    // Predecessor of y[i]
  bsTabC[k+1]=y[i]; // Decrease value for this length IS
  bsTabI[k+1]=i;
}
```

i	1	2	3	4	5	6	7	8	9	10
y_i	60	10	70	80	20	10	30	40	70	20
C										
j										
1										
2										
3										
4										
5										

Strict (`http://ranger.uta.edu/~weems/NOTES2320/LSIS.c`): Similar to monotone with the following exceptions:

2. Determine cost function and base case.

   $C(i) =$ Length of longest *strictly* increasing subsequence ending with y_i.

   $C(0) = 0$

3. Determine general case for cost function.

   $$C(i) = 1 + \max_{j<i \text{ and } y_j<y_i} \{C(j)\}$$ (The *j* that gives $C(i)$ must be saved to allow backtrace.)

Finally, any y_i that is found by `binSearchLast()` will be ignored.

```
for (i=1;i<=n;i++)
{
  // Find SIS that y[i] could be appended to.
  // See CSE 2320 Notes 01 for binSearchLast()
  k=binSearchLast(bsTabC,n+1,y[i]);
  // y[i] only matters if it is not already in table.
  if (bsTabC[k]<y[i]) {
    C[i]=k+1;          // Save length of LIS for y[i]
    j[i]=bsTabI[k];    // Predecessor of y[i]
    bsTabC[k+1]=y[i]; // Decrease value for this length IS
    bsTabI[k+1]=i;
   }
   else
   {
     C[i]=(-1);         // Mark as ignored
     j[i]=(-1);
   }
}
```

i	1	2	3	4	5	6	7	8	9	10
y_i	60	10	70	80	20	10	30	40	70	20
C										
j										

1

2

3

4

5

7.F. SUBSET SUM (`http://ranger.uta.edu/~weems/NOTES2320/subsetSum.c`)

Given a "set" of n positive integer values, find a subset whose sum adds to a value m.

Optimization?

Enumerating subsets (combinations) would take exponential time.

1. Describe problem input. Array $S = S_1, S_2, \ldots, S_n$ and m.

2. Determine cost function and base case.

   $C(i) =$ Smallest index j such that there is some combination of $S_1, S_2, \ldots, S_j$, that includes S_j and sums to i.

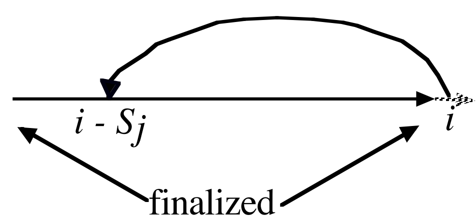
   $C(0) = 0$ (Will assume that $S_0 = 0$)

3. Determine general case for cost function.

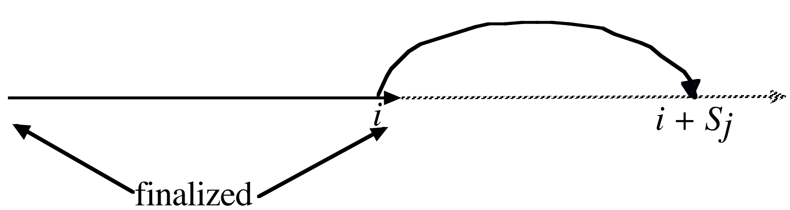
   $$C(i) = \min_{\substack{j \text{ s.t. } C(i - S_j) \text{ is defined} \\ \text{and } C(i - S_j) < j}} \{j\}$$

4. Appropriate ordering of subproblems:

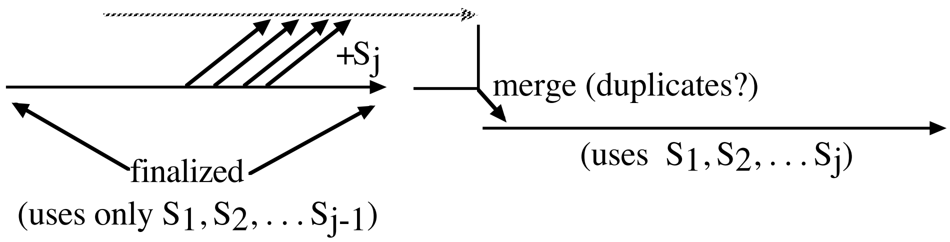
   a. Iterate over i looking backwards (like the cost function) to previous "finalized" solutions.



   b. (Aside, Dijkstra's algorithm-like) Iterate over finalized $C(i)$ to compute $i + S_j$ for each $j > C(i)$ and attempt update forward. After updates, $C(i+1)$ has final value.



c. (Aside) Maintain ordered list of finalized solutions from using $S_1, S_2, \ldots, S_{j-1}$ and generate new ordered list that also uses S_j to reach some new values.



5. Backtrace for solution - if $C(m)$ is defined, then backtrace using C values to subtract out each value in subset. (Indices will appear in strictly decreasing order during backtrace.)

```
// Initialize table for DP
C[0]=0;  // DP base case
// For each potential sum, determine the smallest index such
// that its input value is in a subset to achieve that sum.
for (potentialSum=1; potentialSum<=m; potentialSum ++)
{
  for (j=1;j<=n;j++)
  {
    leftover=potentialSum-S[j];      // To be achieved with other values
    if (leftover>=0 &&               // Possible to have a solution
      C[leftover]<j)                 // Indices are included in
      break;                         // ascending order.
  }
  C[potentialSum]=j;
}

// Output the backtrace
if (C[m]==n+1)
  printf("No solution\n");
else
{
  printf("Solution\n");
  printf("  i   S\n");
  printf("-------\n");
  for (i=m;i>0;i-=S[C[i]])
    printf("%3d %3d\n",C[i],S[C[i]]);
}
```

Example: $m = 12, n = 4$

i	0	1	2	3	4
S_i	0	3	6	7	9

[The S_i values do not require ordering.]

i	0	1	2	3	4	5	6	7	8	9	10	11	12
C_i													

Time is $\Theta(mn)$. Space is $\Theta(m)$. [What happens if m and each S_i are multiplied by the same constant?]