CSE 5311 Notes 4b: van Emde Boas Trees

(Last updated 9/25/15 1:11 PM)

CLRS 20.3 (20.2 on proto vEB is nice, but . . .)

Operations supported in $O(\log\log u)$ time for keys in domain $0 \ldots u-1$, $u = 2^k$:

SEARCH (MEMBER)

INSERT

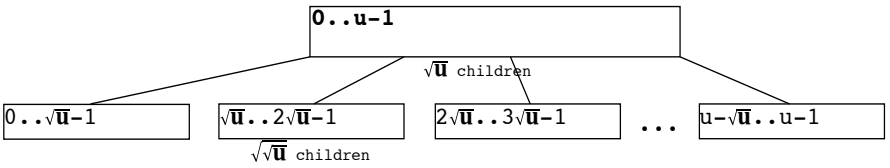
DELETE

MINIMUM - $\Theta(1)$

MAXIMUM - $\Theta(1)$

SUCCESSOR

PREDECESSOR



A simple implementation will use $\Theta(u)$ initialization time and space, but these can be reduced to $\Theta(1)$ and $\Theta(n)$ by using NULL pointers for empty portions of tree and replacing bit vectors and pointer tables with hash tables. (See CLRS problem 20-1.)

	Subtree Domain Size	# of Children Subtree Root	Domain Size for Child Subtrees
	2^{2k}	2^k	2^k
	2^{2k+1}	2^{k+1}	2^k
Example:	2^{22}	2^{11}	2^{11}
	2^{11}	2^6	2^5
	2^5	2^3	2^2
	2^2	2^1	2^1
	2^1 (leaves)		

⇑

High-order bits for SEARCH in $\Theta(\log\log u)$

(Can also line-up leaves contiguously for $\Theta(1)$)

Also, see high, low, and index functions in CLRS, p. 546.

Each node includes the *min* and *max* used within the subtree, to give $\Theta(1)$ MINIMUM and MAXIMUM, but these also help (timewise) with SUCCESSOR and PREDECESSOR.

> If a subtree has no members, then *min* = *max* = NIL (/).
>
> If a subtree has one member, then *min* = *max* and no members are stored below the subtree root.
>
> If a subtree has exactly two members, then *min* < *max*, and only members > *min* are stored below the subtree root. (To repeat, *min* is *not stored* in any of the clusters below.)

Even with *min* and *max*, some SUCCESSOR/ PREDECESSOR requests lead to scans of children, so summary structures are added. These are recursive, so there are *summaries within summaries*.

Concept: When a tree's domain size 2^{2k} (or 2^{2k+1}) is larger than 2^1, there is a *summary structure* of domain size 2^k (or 2^{k+1}) that will have *i* as a member if and only if sub-cluster *i* of the tree has at least one member.

(CLRS, p. 548 - Figure 20.6, members are {2, 3, 4, 5, 7, 14, 15})

```
root (base 0) u 16 min 2 (2) max 15
    summary (base 0) u 4 min 0 (0) max 3
        summary (base 0) u 2 min 0 (0) max 1 (1)
        cluster[0] (base 0) u 2 min 1 (1) max 1
        cluster[1] (base 2) u 2 min 1 (3) max 1
    cluster[0] (base 0) u 4 min 3 (3) max 3
        summary (base 0) u 2 min / max /
        cluster[0] (base 0) u 2 min / max /
        cluster[1] (base 2) u 2 min / max /
    cluster[1] (base 4) u 4 min 0 (4) max 3
        summary (base 0) u 2 min 0 (0) max 1 (1)
        cluster[0] (base 4) u 2 min 1 (5) max 1
        cluster[1] (base 6) u 2 min 1 (7) max 1
    cluster[2] (base 8) u 4 min / max /
        summary (base 0) u 2 min / max /
        cluster[0] (base 8) u 2 min / max /
        cluster[1] (base 10) u 2 min / max /
    cluster[3] (base 12) u 4 min 2 (14) max 3
        summary (base 0) u 2 min 1 (1) max 1
        cluster[0] (base 12) u 2 min / max /
        cluster[1] (base 14) u 2 min 1 (15) max 1
```

Later diagrams will not show empty, untouched clusters . . .

Diagrams were produced using `real.book.c` and `real.driver.c` on course webpage.

Observe that "When an operation traverses this data structure, it will spend a constant amount of time at each level . . ." (CLRS, p. 567)

A larger example: {2, 4, 7, 10, 11, 12, 13, 14, 15, 20, 26, 45, 46, 47, 48, 49, 50, 52, 54, 56, 57, 58, 59}

```
root (base 0) u 64 min 2 (2) max 59 [1: minimum=2]
    summary (base 0) u 8 min 0 (0) max 7
        summary (base 0) u 4 min 0 (0) max 3
            summary (base 0) u 2 min 0 (0) max 1 (1)
            cluster[0] (base 0) u 2 min 1 (1) max 1
            cluster[1] (base 2) u 2 min 0 (2) max 1 (3)
        cluster[0] (base 0) u 2 min 1 (1) max 1
        cluster[1] (base 2) u 2 min 0 (2) max 1 (3)
        cluster[2] (base 4) u 2 min 1 (5) max 1
        cluster[3] (base 6) u 2 min 0 (6) max 1 (7)
    cluster[0] (base 0) u 8 min 4 (4) max 7
        summary (base 0) u 4 min 3 (3) max 3
        cluster[3] (base 6) u 2 min 1 (7) max 1
    cluster[1] (base 8) u 8 min 2 (10) max 7
        summary (base 0) u 4 min 1 (1) max 3
            summary (base 0) u 2 min 1 (1) max 1
            cluster[1] (base 2) u 2 min 0 (2) max 1 (3)
        cluster[1] (base 10) u 2 min 1 (11) max 1
        cluster[2] (base 12) u 2 min 0 (12) max 1 (13)
        cluster[3] (base 14) u 2 min 0 (14) max 1 (15)
    cluster[2] (base 16) u 8 min 4 (20) max 4
    cluster[3] (base 24) u 8 min 2 (26) max 2
    cluster[5] (base 40) u 8 min 5 (45) max 7
        summary (base 0) u 4 min 3 (3) max 3
        cluster[3] (base 46) u 2 min 0 (46) max 1 (47)
    cluster[6] (base 48) u 8 min 0 (48) max 6
        summary (base 0) u 4 min 0 (0) max 3
            summary (base 0) u 2 min 0 (0) max 1 (1)
            cluster[0] (base 0) u 2 min 1 (1) max 1
            cluster[1] (base 2) u 2 min 0 (2) max 1 (3)
        cluster[0] (base 48) u 2 min 1 (49) max 1
        cluster[1] (base 50) u 2 min 0 (50) max 0
        cluster[2] (base 52) u 2 min 0 (52) max 0
        cluster[3] (base 54) u 2 min 0 (54) max 0
    cluster[7] (base 56) u 8 min 0 (56) max 3
        summary (base 0) u 4 min 0 (0) max 1
            summary (base 0) u 2 min 0 (0) max 0
            cluster[0] (base 0) u 2 min 1 (1) max 1
        cluster[0] (base 56) u 2 min 1 (57) max 1
        cluster[1] (base 58) u 2 min 0 (58) max 1 (59)
```

Also:

```
root (base 0) u 64 min 2 (2) max 59 [1: maximum=59]
.
.
.
```

50 (110010) is a member by touching three nodes:

```
root (base 0) u 64 min 2 (2) max 59 [1: member(50)=member(cluster[6],2)]
    summary (base 0) u 8 min 0 (0) max 7
        summary (base 0) u 4 min 0 (0) max 3
            summary (base 0) u 2 min 0 (0) max 1 (1)
            cluster[0] (base 0) u 2 min 1 (1) max 1
            cluster[1] (base 2) u 2 min 0 (2) max 1 (3)
        cluster[0] (base 0) u 2 min 1 (1) max 1
        cluster[1] (base 2) u 2 min 0 (2) max 1 (3)
        cluster[2] (base 4) u 2 min 1 (5) max 1
        cluster[3] (base 6) u 2 min 0 (6) max 1 (7)
    cluster[0] (base 0) u 8 min 4 (4) max 7
        summary (base 0) u 4 min 3 (3) max 3
        cluster[3] (base 6) u 2 min 1 (7) max 1
    cluster[1] (base 8) u 8 min 2 (10) max 7
        summary (base 0) u 4 min 1 (1) max 3
            summary (base 0) u 2 min 1 (1) max 1
            cluster[1] (base 2) u 2 min 0 (2) max 1 (3)
        cluster[1] (base 10) u 2 min 1 (11) max 1
        cluster[2] (base 12) u 2 min 0 (12) max 1 (13)
        cluster[3] (base 14) u 2 min 0 (14) max 1 (15)
    cluster[2] (base 16) u 8 min 4 (20) max 4
    cluster[3] (base 24) u 8 min 2 (26) max 2
    cluster[5] (base 40) u 8 min 5 (45) max 7
        summary (base 0) u 4 min 3 (3) max 3
        cluster[3] (base 46) u 2 min 0 (46) max 1 (47)
    cluster[6] (base 48) u 8 min 0 (48) max 6 [2: member(2)=member(cluster[1],0)]
        summary (base 0) u 4 min 0 (0) max 3
            summary (base 0) u 2 min 0 (0) max 1 (1)
            cluster[0] (base 0) u 2 min 1 (1) max 1
            cluster[1] (base 2) u 2 min 0 (2) max 1 (3)
        cluster[0] (base 48) u 2 min 1 (49) max 1
        cluster[1] (base 50) u 2 min 0 (50) max 0 [3: member(0)=1]
        cluster[2] (base 52) u 2 min 0 (52) max 0
        cluster[3] (base 54) u 2 min 0 (54) max 0
    cluster[7] (base 56) u 8 min 0 (56) max 3
        summary (base 0) u 4 min 0 (0) max 1
            summary (base 0) u 2 min 0 (0) max 0
            cluster[0] (base 0) u 2 min 1 (1) max 1
        cluster[0] (base 56) u 2 min 1 (57) max 1
        cluster[1] (base 58) u 2 min 0 (58) max 1 (59)
```

Successor of 0 is 2:

```
root (base 0) u 64 min 2 (2) max 59 [1: root successor(0)=V->min=2]
```

Successor of 49 (110001) is 50:

```
root (base 0) u 64 min 2 (2) max 59 [1: successor(49)
                                     2: must descend
                                     4: maxLow=6
                                     20: descent successor(49)=50]
    summary (base 0) u 8 min 0 (0) max 7
        summary (base 0) u 4 min 0 (0) max 3
            summary (base 0) u 2 min 0 (0) max 1 (1)
            cluster[0] (base 0) u 2 min 1 (1) max 1
            cluster[1] (base 2) u 2 min 0 (2) max 1 (3)
        cluster[0] (base 0) u 2 min 1 (1) max 1
        cluster[1] (base 2) u 2 min 0 (2) max 1 (3)
        cluster[2] (base 4) u 2 min 1 (5) max 1
        cluster[3] (base 6) u 2 min 0 (6) max 1 (7)
    cluster[0] (base 0) u 8 min 4 (4) max 7
        summary (base 0) u 4 min 3 (3) max 3
        cluster[3] (base 6) u 2 min 1 (7) max 1
    cluster[1] (base 8) u 8 min 2 (10) max 7
        summary (base 0) u 4 min 1 (1) max 3
            summary (base 0) u 2 min 1 (1) max 1
            cluster[1] (base 2) u 2 min 0 (2) max 1 (3)
        cluster[1] (base 10) u 2 min 1 (11) max 1
        cluster[2] (base 12) u 2 min 0 (12) max 1 (13)
        cluster[3] (base 14) u 2 min 0 (14) max 1 (15)
    cluster[2] (base 16) u 8 min 4 (20) max 4
    cluster[3] (base 24) u 8 min 2 (26) max 2
    cluster[5] (base 40) u 8 min 5 (45) max 7
        summary (base 0) u 4 min 3 (3) max 3
        cluster[3] (base 46) u 2 min 0 (46) max 1 (47)
    cluster[6] (base 48) u 8 min 0 (48) max 6 [3: maximum=6
                                               5: successor(1)
                                               6: must descend
                                               8: maxLow=1
                                               9: descent failed
                                               19: neighbor successor(1)=2]
        summary (base 0) u 4 min 0 (0) max 3 [10: successor(0)
                                              11: must descend
                                              13: maxLow=1
                                              16: descent successor(0)=1]
            summary (base 0) u 2 min 0 (0) max 1 (1)
            cluster[0] (base 0) u 2 min 1 (1) max 1 [12: maximum=1
                                                     14: successor(0)
                                                     15: leaf successor(0)=1]
            cluster[1] (base 2) u 2 min 0 (2) max 1 (3)
        cluster[0] (base 48) u 2 min 1 (49) max 1 [7: maximum=1]
        cluster[1] (base 50) u 2 min 0 (50) max 0 [17: use succCluster
                                                   18: minimum=0]
        cluster[2] (base 52) u 2 min 0 (52) max 0
        cluster[3] (base 54) u 2 min 0 (54) max 0
    cluster[7] (base 56) u 8 min 0 (56) max 3
        summary (base 0) u 4 min 0 (0) max 1
            summary (base 0) u 2 min 0 (0) max 0
            cluster[0] (base 0) u 2 min 1 (1) max 1
        cluster[0] (base 56) u 2 min 1 (57) max 1
        cluster[1] (base 58) u 2 min 0 (58) max 1 (59)
```

Successor of 26 (011010) is 45:

```
root (base 0) u 64 min 2 (2) max 59 [1: successor(26)
                                     2: must descend
                                     4: maxLow=2
                                     5: descent failed
                                     26: neighbor successor(26)=45]
    summary (base 0) u 8 min 0 (0) max 7 [6: successor(3)
                                          7: must descend
                                          9: maxLow=1
                                          10: descent failed
                                          23: neighbor successor(3)=5]
        summary (base 0) u 4 min 0 (0) max 3 [11: successor(1)
                                              12: must descend
                                              14: maxLow=1
                                              15: descent failed
                                              20: neighbor successor(1)=2]
            summary (base 0) u 2 min 0 (0) max 1 (1) [16: successor(0)
                                                      17: leaf successor(0)=1]
            cluster[0] (base 0) u 2 min 1 (1) max 1 [13: maximum=1]
            cluster[1] (base 2) u 2 min 0 (2) max 1 (3) [18: use succCluster
                                                         19: minimum=0]
        cluster[0] (base 0) u 2 min 1 (1) max 1
        cluster[1] (base 2) u 2 min 0 (2) max 1 (3) [8: maximum=1]
        cluster[2] (base 4) u 2 min 1 (5) max 1 [21: use succCluster
                                                 22: minimum=1]
        cluster[3] (base 6) u 2 min 0 (6) max 1 (7)
    cluster[0] (base 0) u 8 min 4 (4) max 7
        summary (base 0) u 4 min 3 (3) max 3
        cluster[3] (base 6) u 2 min 1 (7) max 1
    cluster[1] (base 8) u 8 min 2 (10) max 7
        summary (base 0) u 4 min 1 (1) max 3
            summary (base 0) u 2 min 1 (1) max 1
            cluster[1] (base 2) u 2 min 0 (2) max 1 (3)
        cluster[1] (base 10) u 2 min 1 (11) max 1
        cluster[2] (base 12) u 2 min 0 (12) max 1 (13)
        cluster[3] (base 14) u 2 min 0 (14) max 1 (15)
    cluster[2] (base 16) u 8 min 4 (20) max 4
    cluster[3] (base 24) u 8 min 2 (26) max 2 [3: maximum=2]
    cluster[5] (base 40) u 8 min 5 (45) max 7 [24: use succCluster
                                               25: minimum=5]
        summary (base 0) u 4 min 3 (3) max 3
        cluster[3] (base 46) u 2 min 0 (46) max 1 (47)
    cluster[6] (base 48) u 8 min 0 (48) max 6
        summary (base 0) u 4 min 0 (0) max 3
            summary (base 0) u 2 min 0 (0) max 1 (1)
            cluster[0] (base 0) u 2 min 1 (1) max 1
            cluster[1] (base 2) u 2 min 0 (2) max 1 (3)
        cluster[0] (base 48) u 2 min 1 (49) max 1
        cluster[1] (base 50) u 2 min 0 (50) max 0
        cluster[2] (base 52) u 2 min 0 (52) max 0
        cluster[3] (base 54) u 2 min 0 (54) max 0
    cluster[7] (base 56) u 8 min 0 (56) max 3
        summary (base 0) u 4 min 0 (0) max 1
            summary (base 0) u 2 min 0 (0) max 0
            cluster[0] (base 0) u 2 min 1 (1) max 1
        cluster[0] (base 56) u 2 min 1 (57) max 1
        cluster[1] (base 58) u 2 min 0 (58) max 1 (59)
```

Predecessor of 58 (111010) is 57:

```
root (base 0) u 64 min 2 (2) max 59 [1: predecessor(58)
                                     2: must descend
                                     4: minLow=0
                                     21: descent predecessor(58)=57]
    summary (base 0) u 8 min 0 (0) max 7
        summary (base 0) u 4 min 0 (0) max 3
            summary (base 0) u 2 min 0 (0) max 1 (1)
            cluster[0] (base 0) u 2 min 1 (1) max 1
            cluster[1] (base 2) u 2 min 0 (2) max 1 (3)
        cluster[0] (base 0) u 2 min 1 (1) max 1
        cluster[1] (base 2) u 2 min 0 (2) max 1 (3)
        cluster[2] (base 4) u 2 min 1 (5) max 1
        cluster[3] (base 6) u 2 min 0 (6) max 1 (7)
    cluster[0] (base 0) u 8 min 4 (4) max 7
        summary (base 0) u 4 min 3 (3) max 3
        cluster[3] (base 6) u 2 min 1 (7) max 1
    cluster[1] (base 8) u 8 min 2 (10) max 7
        summary (base 0) u 4 min 1 (1) max 3
            summary (base 0) u 2 min 1 (1) max 1
            cluster[1] (base 2) u 2 min 0 (2) max 1 (3)
        cluster[1] (base 10) u 2 min 1 (11) max 1
        cluster[2] (base 12) u 2 min 0 (12) max 1 (13)
        cluster[3] (base 14) u 2 min 0 (14) max 1 (15)
    cluster[2] (base 16) u 8 min 4 (20) max 4
    cluster[3] (base 24) u 8 min 2 (26) max 2
    cluster[5] (base 40) u 8 min 5 (45) max 7
        summary (base 0) u 4 min 3 (3) max 3
        cluster[3] (base 46) u 2 min 0 (46) max 1 (47)
    cluster[6] (base 48) u 8 min 0 (48) max 6
        summary (base 0) u 4 min 0 (0) max 3
            summary (base 0) u 2 min 0 (0) max 1 (1)
            cluster[0] (base 0) u 2 min 1 (1) max 1
            cluster[1] (base 2) u 2 min 0 (2) max 1 (3)
        cluster[0] (base 48) u 2 min 1 (49) max 1
        cluster[1] (base 50) u 2 min 0 (50) max 0
        cluster[2] (base 52) u 2 min 0 (52) max 0
        cluster[3] (base 54) u 2 min 0 (54) max 0
    cluster[7] (base 56) u 8 min 0 (56) max 3 [3: minimum=0
                                               5: predecessor(2)
                                               6: must descend
                                               8: minLow=0
                                               9: descent failed
                                               20: neighbor predecessor(2)=1]
        summary (base 0) u 4 min 0 (0) max 1 [10: predecessor(1)
                                              11: must descend
                                              13: minLow=1
                                              14: descent failed
                                              17: special - predecessor=V->min=0]
            summary (base 0) u 2 min 0 (0) max 0 [15: predecessor(0)
                                                  16: no leaf predecessor(0)
                                                      =vEBNIL]
            cluster[0] (base 0) u 2 min 1 (1) max 1 [12: minimum=1]
        cluster[0] (base 56) u 2 min 1 (57) max 1 [18: use predCluster
                                                   19: maximum=1]
        cluster[1] (base 58) u 2 min 0 (58) max 1 (59) [7: minimum=0]
```

Predecessor of 49 (110001) is 48:

```
root (base 0) u 64 min 2 (2) max 59 [1: predecessor(49)
                                     2: must descend
                                     4: minLow=0
                                     19: descent predecessor(49)=48]
    summary (base 0) u 8 min 0 (0) max 7
        summary (base 0) u 4 min 0 (0) max 3
            summary (base 0) u 2 min 0 (0) max 1 (1)
            cluster[0] (base 0) u 2 min 1 (1) max 1
            cluster[1] (base 2) u 2 min 0 (2) max 1 (3)
        cluster[0] (base 0) u 2 min 1 (1) max 1
        cluster[1] (base 2) u 2 min 0 (2) max 1 (3)
        cluster[2] (base 4) u 2 min 1 (5) max 1
        cluster[3] (base 6) u 2 min 0 (6) max 1 (7)
    cluster[0] (base 0) u 8 min 4 (4) max 7
        summary (base 0) u 4 min 3 (3) max 3
        cluster[3] (base 6) u 2 min 1 (7) max 1
    cluster[1] (base 8) u 8 min 2 (10) max 7
        summary (base 0) u 4 min 1 (1) max 3
            summary (base 0) u 2 min 1 (1) max 1
            cluster[1] (base 2) u 2 min 0 (2) max 1 (3)
        cluster[1] (base 10) u 2 min 1 (11) max 1
        cluster[2] (base 12) u 2 min 0 (12) max 1 (13)
        cluster[3] (base 14) u 2 min 0 (14) max 1 (15)
    cluster[2] (base 16) u 8 min 4 (20) max 4
    cluster[3] (base 24) u 8 min 2 (26) max 2
    cluster[5] (base 40) u 8 min 5 (45) max 7
        summary (base 0) u 4 min 3 (3) max 3
        cluster[3] (base 46) u 2 min 0 (46) max 1 (47)
    cluster[6] (base 48) u 8 min 0 (48) max 6 [3: minimum=0
                                               5: predecessor(1)
                                               6: must descend
                                               8: minLow=1
                                               9: descent failed
                                               18: special - predecessor=V->min=0]
        summary (base 0) u 4 min 0 (0) max 3 [10: predecessor(0)
                                              11: must descend
                                              13: minLow=1
                                              14: descent failed
                                              17: no neighbor predecessor(0)
                                                  =vEBNIL]
            summary (base 0) u 2 min 0 (0) max 1 (1) [15: predecessor(0)
                                                      16: no leaf predecessor(0)
                                                          =vEBNIL]
            cluster[0] (base 0) u 2 min 1 (1) max 1 [12: minimum=1]
            cluster[1] (base 2) u 2 min 0 (2) max 1 (3)
        cluster[0] (base 48) u 2 min 1 (49) max 1 [7: minimum=1]
        cluster[1] (base 50) u 2 min 0 (50) max 0
        cluster[2] (base 52) u 2 min 0 (52) max 0
        cluster[3] (base 54) u 2 min 0 (54) max 0
    cluster[7] (base 56) u 8 min 0 (56) max 3
        summary (base 0) u 4 min 0 (0) max 1
            summary (base 0) u 2 min 0 (0) max 0
            cluster[0] (base 0) u 2 min 1 (1) max 1
        cluster[0] (base 56) u 2 min 1 (57) max 1
        cluster[1] (base 58) u 2 min 0 (58) max 1 (59)
```

Insert of 42 (101010) completed:

```
root (base 0) u 64 min 2 (2) max 59 [1: insert(42)]
    summary (base 0) u 8 min 0 (0) max 7
        summary (base 0) u 4 min 0 (0) max 3
            summary (base 0) u 2 min 0 (0) max 1 (1)
            cluster[0] (base 0) u 2 min 1 (1) max 1
            cluster[1] (base 2) u 2 min 0 (2) max 1 (3)
        cluster[0] (base 0) u 2 min 1 (1) max 1
        cluster[1] (base 2) u 2 min 0 (2) max 1 (3)
        cluster[2] (base 4) u 2 min 1 (5) max 1
        cluster[3] (base 6) u 2 min 0 (6) max 1 (7)
    cluster[0] (base 0) u 8 min 4 (4) max 7
        summary (base 0) u 4 min 3 (3) max 3
        cluster[3] (base 6) u 2 min 1 (7) max 1
    cluster[1] (base 8) u 8 min 2 (10) max 7
        summary (base 0) u 4 min 1 (1) max 3
            summary (base 0) u 2 min 1 (1) max 1
            cluster[1] (base 2) u 2 min 0 (2) max 1 (3)
        cluster[1] (base 10) u 2 min 1 (11) max 1
        cluster[2] (base 12) u 2 min 0 (12) max 1 (13)
        cluster[3] (base 14) u 2 min 0 (14) max 1 (15)
    cluster[2] (base 16) u 8 min 4 (20) max 4
    cluster[3] (base 24) u 8 min 2 (26) max 2
    cluster[5] (base 40) u 8 min 2 (42) max 7 [2: minimum=5
                                               3: insert into non-empty sub-cluster
                                               4: insert(2)
                                               5: swapped arg 2 with V->min 5]
        summary (base 0) u 4 min 2 (2) max 3 [8: insert(2)
                                              9: swapped arg 2 with V->min 3]
            summary (base 0) u 2 min 1 (1) max 1 [12: insert(1)
                                                  13: emptyTreeInsert(1)]
            cluster[1] (base 2) u 2 min 1 (3) max 1 [10: minimum=-1
                                                     11: insert into empty sub-
                                                         cluster
                                                     14: emptyTreeInsert(1)]
        cluster[2] (base 44) u 2 min 1 (45) max 1 [6: minimum=-1
                                                   7: insert into empty sub-cluster
                                                   15: emptyTreeInsert(1)]
        cluster[3] (base 46) u 2 min 0 (46) max 1 (47)
    cluster[6] (base 48) u 8 min 0 (48) max 6
        summary (base 0) u 4 min 0 (0) max 3
            summary (base 0) u 2 min 0 (0) max 1 (1)
            cluster[0] (base 0) u 2 min 1 (1) max 1
            cluster[1] (base 2) u 2 min 0 (2) max 1 (3)
        cluster[0] (base 48) u 2 min 1 (49) max 1
        cluster[1] (base 50) u 2 min 0 (50) max 0
        cluster[2] (base 52) u 2 min 0 (52) max 0
        cluster[3] (base 54) u 2 min 0 (54) max 0
    cluster[7] (base 56) u 8 min 0 (56) max 3
        summary (base 0) u 4 min 0 (0) max 1
            summary (base 0) u 2 min 0 (0) max 0
            cluster[0] (base 0) u 2 min 1 (1) max 1
        cluster[0] (base 56) u 2 min 1 (57) max 1
        cluster[1] (base 58) u 2 min 0 (58) max 1 (59)
```

Insert of 55 (110111) completed:

```
root (base 0) u 64 min 2 (2) max 59 [1: insert(55)]
    summary (base 0) u 8 min 0 (0) max 7
        summary (base 0) u 4 min 0 (0) max 3
            summary (base 0) u 2 min 0 (0) max 1 (1)
            cluster[0] (base 0) u 2 min 1 (1) max 1
            cluster[1] (base 2) u 2 min 0 (2) max 1 (3)
        cluster[0] (base 0) u 2 min 1 (1) max 1
        cluster[1] (base 2) u 2 min 0 (2) max 1 (3)
        cluster[2] (base 4) u 2 min 1 (5) max 1
        cluster[3] (base 6) u 2 min 0 (6) max 1 (7)
    cluster[0] (base 0) u 8 min 4 (4) max 7
        summary (base 0) u 4 min 3 (3) max 3
        cluster[3] (base 6) u 2 min 1 (7) max 1
    cluster[1] (base 8) u 8 min 2 (10) max 7
        summary (base 0) u 4 min 1 (1) max 3
            summary (base 0) u 2 min 1 (1) max 1
            cluster[1] (base 2) u 2 min 0 (2) max 1 (3)
        cluster[1] (base 10) u 2 min 1 (11) max 1
        cluster[2] (base 12) u 2 min 0 (12) max 1 (13)
        cluster[3] (base 14) u 2 min 0 (14) max 1 (15)
    cluster[2] (base 16) u 8 min 4 (20) max 4
    cluster[3] (base 24) u 8 min 2 (26) max 2
    cluster[5] (base 40) u 8 min 2 (42) max 7
        summary (base 0) u 4 min 2 (2) max 3
            summary (base 0) u 2 min 1 (1) max 1
            cluster[1] (base 2) u 2 min 1 (3) max 1
        cluster[2] (base 44) u 2 min 1 (45) max 1
        cluster[3] (base 46) u 2 min 0 (46) max 1 (47)
    cluster[6] (base 48) u 8 min 0 (48) max 7 [2: minimum=0
                                               3: insert into non-empty sub-cluster
                                               4: insert(7)
                                               9: increasing V->max to 7]
        summary (base 0) u 4 min 0 (0) max 3
            summary (base 0) u 2 min 0 (0) max 1 (1)
            cluster[0] (base 0) u 2 min 1 (1) max 1
            cluster[1] (base 2) u 2 min 0 (2) max 1 (3)
        cluster[0] (base 48) u 2 min 1 (49) max 1
        cluster[1] (base 50) u 2 min 0 (50) max 0
        cluster[2] (base 52) u 2 min 0 (52) max 0
        cluster[3] (base 54) u 2 min 0 (54) max 1 (55) [5: minimum=0
                                                        6: insert into non-empty
                                                           sub-cluster
                                                        7: insert(1)
                                                        8: increasing V->max to 1]
    cluster[7] (base 56) u 8 min 0 (56) max 3
        summary (base 0) u 4 min 0 (0) max 1
            summary (base 0) u 2 min 0 (0) max 0
            cluster[0] (base 0) u 2 min 1 (1) max 1
        cluster[0] (base 56) u 2 min 1 (57) max 1
        cluster[1] (base 58) u 2 min 0 (58) max 1 (59)
```

Delete of 42 (101010) completed:

```
root (base 0) u 64 min 2 (2) max 59 [1: delete(42)
                                     2: deleting 42 from sub-cluster]
    summary (base 0) u 8 min 0 (0) max 7
        summary (base 0) u 4 min 0 (0) max 3
            summary (base 0) u 2 min 0 (0) max 1 (1)
            cluster[0] (base 0) u 2 min 1 (1) max 1
            cluster[1] (base 2) u 2 min 0 (2) max 1 (3)
        cluster[0] (base 0) u 2 min 1 (1) max 1
        cluster[1] (base 2) u 2 min 0 (2) max 1 (3)
        cluster[2] (base 4) u 2 min 1 (5) max 1
        cluster[3] (base 6) u 2 min 0 (6) max 1 (7)
    cluster[0] (base 0) u 8 min 4 (4) max 7
        summary (base 0) u 4 min 3 (3) max 3
        cluster[3] (base 6) u 2 min 1 (7) max 1
    cluster[1] (base 8) u 8 min 2 (10) max 7
        summary (base 0) u 4 min 1 (1) max 3
            summary (base 0) u 2 min 1 (1) max 1
            cluster[1] (base 2) u 2 min 0 (2) max 1 (3)
        cluster[1] (base 10) u 2 min 1 (11) max 1
        cluster[2] (base 12) u 2 min 0 (12) max 1 (13)
        cluster[3] (base 14) u 2 min 0 (14) max 1 (15)
    cluster[2] (base 16) u 8 min 4 (20) max 4
    cluster[3] (base 24) u 8 min 2 (26) max 2
    cluster[5] (base 40) u 8 min 5 (45) max 7 [3: delete(2)
                                               4: cluster losing minimum
                                               7: V->min replaced by 5
                                               8: deleting 5 from sub-cluster
                                               12: sub-cluster for 5 is empty
                                               28: minimum=5]
        summary (base 0) u 4 min 3 (3) max 3 [5: minimum=2
                                              13: delete(2)
                                              14: cluster losing minimum
                                              17: V->min replaced by 3
                                              18: deleting 3 from sub-cluster
                                              22: sub-cluster for 3 is empty
                                              25: need new max to replace 3
                                              27: new max not found, now a one-element
                                                  cluster]
            summary (base 0) u 2 min / max / [15: minimum=1
                                              23: delete(1)
                                              24: cluster is losing its one member
                                              26: maximum=-1]
            cluster[1] (base 2) u 2 min / max / [16: minimum=1
                                                 19: delete(1)
                                                 20: cluster is losing its one member
                                                 21: minimum=-1]
        cluster[2] (base 44) u 2 min / max / [6: minimum=1
                                              9: delete(1)
                                              10: cluster is losing its one member
                                              11: minimum=-1]
        cluster[3] (base 46) u 2 min 0 (46) max 1 (47)
    cluster[6] (base 48) u 8 min 0 (48) max 7
        summary (base 0) u 4 min 0 (0) max 3
            summary (base 0) u 2 min 0 (0) max 1 (1)
            cluster[0] (base 0) u 2 min 1 (1) max 1
            cluster[1] (base 2) u 2 min 0 (2) max 1 (3)
        cluster[0] (base 48) u 2 min 1 (49) max 1
        cluster[1] (base 50) u 2 min 0 (50) max 0
        cluster[2] (base 52) u 2 min 0 (52) max 0
        cluster[3] (base 54) u 2 min 0 (54) max 1 (55)
    cluster[7] (base 56) u 8 min 0 (56) max 3
        summary (base 0) u 4 min 0 (0) max 1
            summary (base 0) u 2 min 0 (0) max 0
            cluster[0] (base 0) u 2 min 1 (1) max 1
        cluster[0] (base 56) u 2 min 1 (57) max 1
        cluster[1] (base 58) u 2 min 0 (58) max 1 (59)
```

Delete of 55 (110111) completed:

```
root (base 0) u 64 min 2 (2) max 59 [1: delete(55)
                                     2: deleting 55 from sub-cluster]
    summary (base 0) u 8 min 0 (0) max 7
        summary (base 0) u 4 min 0 (0) max 3
            summary (base 0) u 2 min 0 (0) max 1 (1)
            cluster[0] (base 0) u 2 min 1 (1) max 1
            cluster[1] (base 2) u 2 min 0 (2) max 1 (3)
        cluster[0] (base 0) u 2 min 1 (1) max 1
        cluster[1] (base 2) u 2 min 0 (2) max 1 (3)
        cluster[2] (base 4) u 2 min 1 (5) max 1
        cluster[3] (base 6) u 2 min 0 (6) max 1 (7)
    cluster[0] (base 0) u 8 min 4 (4) max 7
        summary (base 0) u 4 min 3 (3) max 3
        cluster[3] (base 6) u 2 min 1 (7) max 1
    cluster[1] (base 8) u 8 min 2 (10) max 7
        summary (base 0) u 4 min 1 (1) max 3
            summary (base 0) u 2 min 1 (1) max 1
            cluster[1] (base 2) u 2 min 0 (2) max 1 (3)
        cluster[1] (base 10) u 2 min 1 (11) max 1
        cluster[2] (base 12) u 2 min 0 (12) max 1 (13)
        cluster[3] (base 14) u 2 min 0 (14) max 1 (15)
    cluster[2] (base 16) u 8 min 4 (20) max 4
    cluster[3] (base 24) u 8 min 2 (26) max 2
    cluster[5] (base 40) u 8 min 5 (45) max 7
        summary (base 0) u 4 min 3 (3) max 3
        cluster[3] (base 46) u 2 min 0 (46) max 1 (47)
    cluster[6] (base 48) u 8 min 0 (48) max 6 [3: delete(7)
                                               4: deleting 7 from sub-cluster
                                               9: corrected V->max to 6
                                               10: minimum=0]
        summary (base 0) u 4 min 0 (0) max 3
            summary (base 0) u 2 min 0 (0) max 1 (1)
            cluster[0] (base 0) u 2 min 1 (1) max 1
            cluster[1] (base 2) u 2 min 0 (2) max 1 (3)
        cluster[0] (base 48) u 2 min 1 (49) max 1
        cluster[1] (base 50) u 2 min 0 (50) max 0
        cluster[2] (base 52) u 2 min 0 (52) max 0
        cluster[3] (base 54) u 2 min 0 (54) max 0 [5: delete(1)
                                                   6: leaf cluster going from two
                                                      members to one
                                                   7: minimum=0
                                                   8: maximum=0]
    cluster[7] (base 56) u 8 min 0 (56) max 3
        summary (base 0) u 4 min 0 (0) max 1
            summary (base 0) u 2 min 0 (0) max 0
            cluster[0] (base 0) u 2 min 1 (1) max 1
        cluster[0] (base 56) u 2 min 1 (57) max 1
        cluster[1] (base 58) u 2 min 0 (58) max 1 (59)
```

Delete of 59 (111011) completed:

```
root (base 0) u 64 min 2 (2) max 58 [1: delete(59)
                                     2: deleting 59 from sub-cluster
                                     12: corrected V->max to 58]
    summary (base 0) u 8 min 0 (0) max 7
        summary (base 0) u 4 min 0 (0) max 3
            summary (base 0) u 2 min 0 (0) max 1 (1)
            cluster[0] (base 0) u 2 min 1 (1) max 1
            cluster[1] (base 2) u 2 min 0 (2) max 1 (3)
        cluster[0] (base 0) u 2 min 1 (1) max 1
        cluster[1] (base 2) u 2 min 0 (2) max 1 (3)
        cluster[2] (base 4) u 2 min 1 (5) max 1
        cluster[3] (base 6) u 2 min 0 (6) max 1 (7)
    cluster[0] (base 0) u 8 min 4 (4) max 7
        summary (base 0) u 4 min 3 (3) max 3
        cluster[3] (base 6) u 2 min 1 (7) max 1
    cluster[1] (base 8) u 8 min 2 (10) max 7
        summary (base 0) u 4 min 1 (1) max 3
            summary (base 0) u 2 min 1 (1) max 1
            cluster[1] (base 2) u 2 min 0 (2) max 1 (3)
        cluster[1] (base 10) u 2 min 1 (11) max 1
        cluster[2] (base 12) u 2 min 0 (12) max 1 (13)
        cluster[3] (base 14) u 2 min 0 (14) max 1 (15)
    cluster[2] (base 16) u 8 min 4 (20) max 4
    cluster[3] (base 24) u 8 min 2 (26) max 2
    cluster[5] (base 40) u 8 min 5 (45) max 7
        summary (base 0) u 4 min 3 (3) max 3
        cluster[3] (base 46) u 2 min 0 (46) max 1 (47)
    cluster[6] (base 48) u 8 min 0 (48) max 6
        summary (base 0) u 4 min 0 (0) max 3
            summary (base 0) u 2 min 0 (0) max 1 (1)
            cluster[0] (base 0) u 2 min 1 (1) max 1
            cluster[1] (base 2) u 2 min 0 (2) max 1 (3)
        cluster[0] (base 48) u 2 min 1 (49) max 1
        cluster[1] (base 50) u 2 min 0 (50) max 0
        cluster[2] (base 52) u 2 min 0 (52) max 0
        cluster[3] (base 54) u 2 min 0 (54) max 0
    cluster[7] (base 56) u 8 min 0 (56) max 2 [3: delete(3)
                                               4: deleting 3 from sub-cluster
                                               9: corrected V->max to 2
                                               10: minimum=0
                                               11: maximum=2]
        summary (base 0) u 4 min 0 (0) max 1
            summary (base 0) u 2 min 0 (0) max 0
            cluster[0] (base 0) u 2 min 1 (1) max 1
        cluster[0] (base 56) u 2 min 1 (57) max 1
        cluster[1] (base 58) u 2 min 0 (58) max 0 [5: delete(1)
                                                   6: leaf cluster going from two
                                                      members to one
                                                   7: minimum=0
                                                   8: maximum=0]
```

Other bit-twiddling:

https://graphics.stanford.edu/~seander/bithacks.html

Avoiding initialization: P. 37 of http://dl.acm.org.ezproxy.uta.edu/citation.cfm?doid=2597757.2535933

Test 1, Summer 2010

Fill in the `min` and `max` blanks for the following instance of a van Emde Boas tree for the set {0, 1, 8, 10, 11, 12, 13}. You should give these as values in the local universe (`0..u-1`). Instead of using the symbol "/" for NIL, use the symbol "∅". (10 points)

```
root (base 0) u 16 min _______ max _______

    summary (base 0) u 4 min _______ max _______

        summary (base 0) u 2 min _______ max _______

        cluster[0] (base 0) u 2 min _______ max _______

        cluster[1] (base 2) u 2 min _______ max _______

    cluster[0] (base 0) u 4 min _______ max _______

        summary (base 0) u 2 min _______ max _______

        cluster[0] (base 0) u 2 min _______ max _______

        cluster[1] (base 2) u 2 min _______ max _______

    cluster[1] (base 4) u 4 min _______ max _______

        summary (base 0) u 2 min _______ max _______

        cluster[0] (base 4) u 2 min _______ max _______

        cluster[1] (base 6) u 2 min _______ max _______

    cluster[2] (base 8) u 4 min _______ max _______

        summary (base 0) u 2 min _______ max _______

        cluster[0] (base 8) u 2 min _______ max _______

        cluster[1] (base 10) u 2 min _______ max _______

    cluster[3] (base 12) u 4 min _______ max _______

        summary (base 0) u 2 min _______ max _______

        cluster[0] (base 12) u 2 min _______ max _______

        cluster[1] (base 14) u 2 min _______ max _______
```