CSE 5311 Notes 7: Priority Queues

(Last updated 6/8/08 8:46 AM)

Chart on p. 456, CLRS (binary, binomial, Fibonacci heaps)

MAKE-HEAP

INSERT

MINIMUM

EXTRACT-MIN

UNION (MELD/MERGE)

DECREASE-KEY

DELETE

Applications - sorting (?), scheduling, greedy algorithms, discrete event simulation

Ordered lists - Suitable if n is extremely small (some simulations)

Binary trees - O(log n) operations, but larger constant than binary heaps. O(n) for UNION.

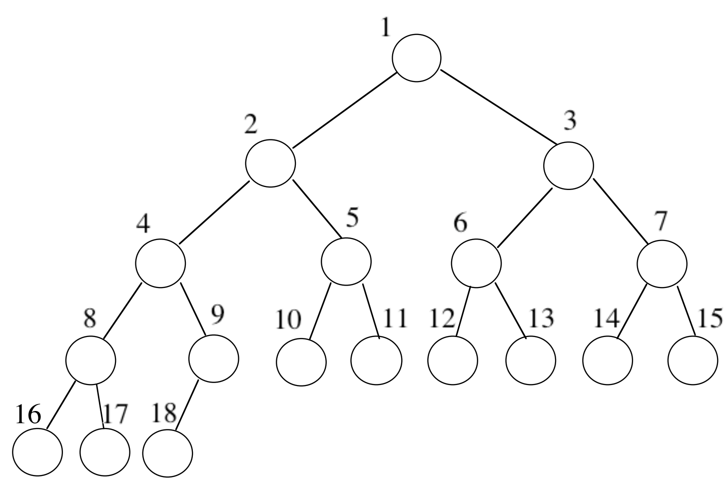
Binary heap (review)

Conceptual structure

Ordering criteria

Mapping to table

O(log n) operations, except UNION



d-heap

Generalizes binary heap with fan-out of d to get shallower structure.

Similar details as binary heap for mapping to an array.

Useful when many DECREASE-KEYs occur (example: Prim's MST, $\Theta(|E|\log|V|)$ - use d = |E| /|V|)

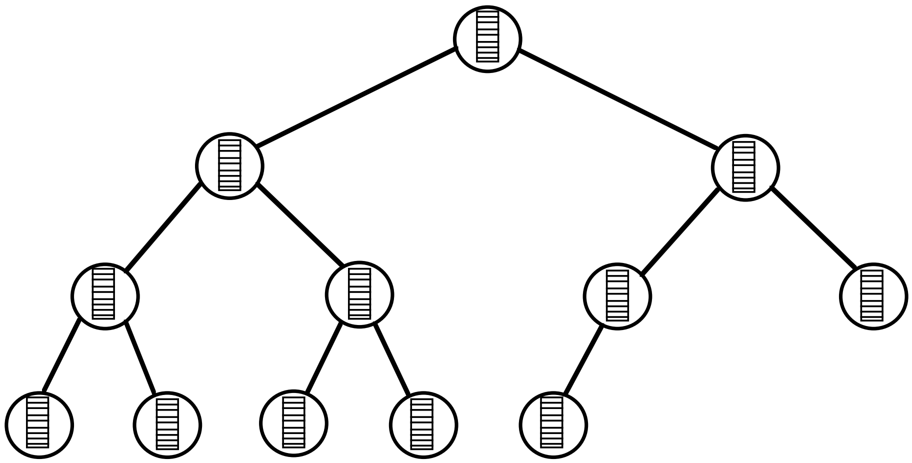
General issue - single-valued nodes vs. nodes containing table ("sack") with O(log n) values

Table is in ascending priority order.

Most operations operate locally on one table.

If the first table element changes (minheap), then traditional heap processing occurs.

Tree structure must be linked, i.e. mapping nodes to table is too slow.



LEFTIST HEAPS

Binary tree, heap ordered

Each node has *null path length*

Either subtree empty ⇒ NPL = 0

Otherwise NPL = 1 + min(left→NPL, right→NPL) (Empty tree - view NPL as -1)

Leftist property: left→NPL ≥ right→NPL at all nodes.

Leftmost path length is O(n).

Rightmost path length is O(log n).

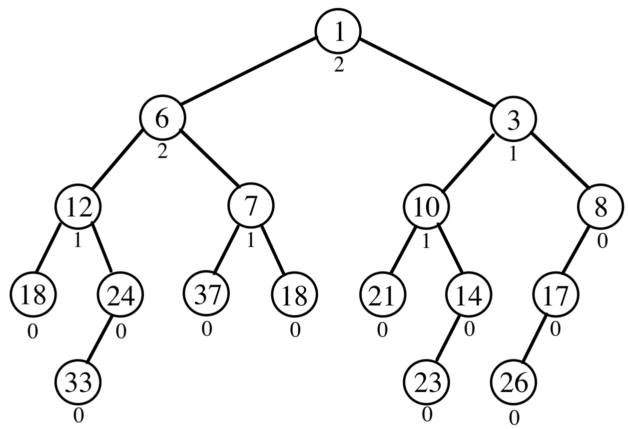
Leftist tree with r nodes on right path must have at least 2^r - 1 nodes.

(e.g. NPL is height of maximum embedded complete binary tree)

Operations take O(log n) by avoiding left paths and emphasizing right paths.

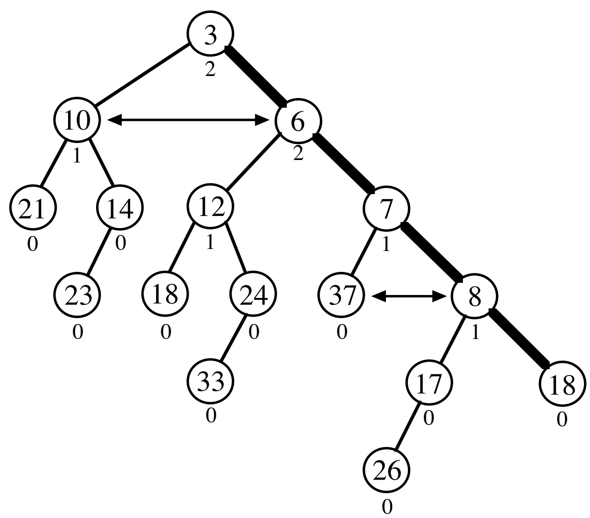
Occasionally, left and right subtrees must be swapped.

Example: EXTRACT-MIN

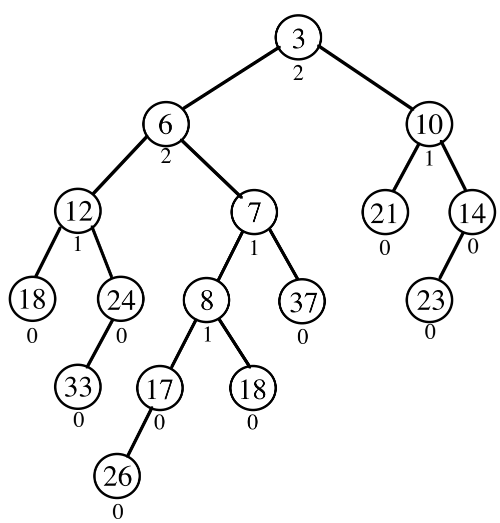


Root has item to return.

Recursively, merge right paths of two subheaps (top-down) keeping same left children.



Swap subtrees, bottom-up, if necessary to restore leftist property.



UNION takes O(log *n*) time, so use to implement other operations.

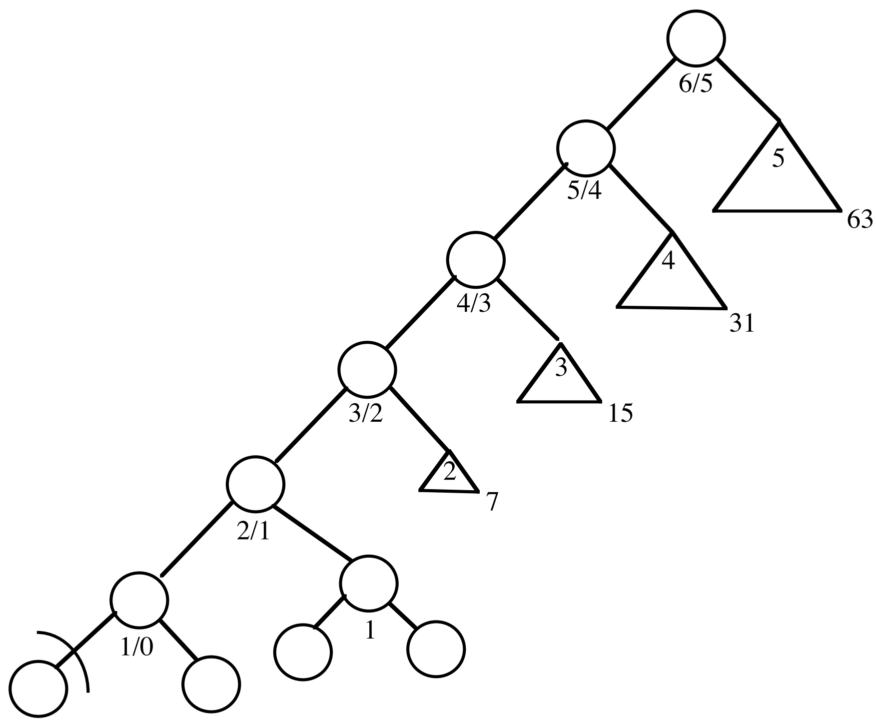
BUT, DECREASE-KEY may involve a node $\Omega(n)$ away from root, so swapping through ancestors is too slow.

1. Find node X via another data structure.

2. *Cut* X's subtree away from parent.

3. Update NPL on former ancestors of X, swapping subtrees to restore leftist property.

   Decrease in NPL continues to propagate only when ancestor NPL decreases.
   (Implies O(log *n*)) Ancestors in diagram are marked with before/after NPL.

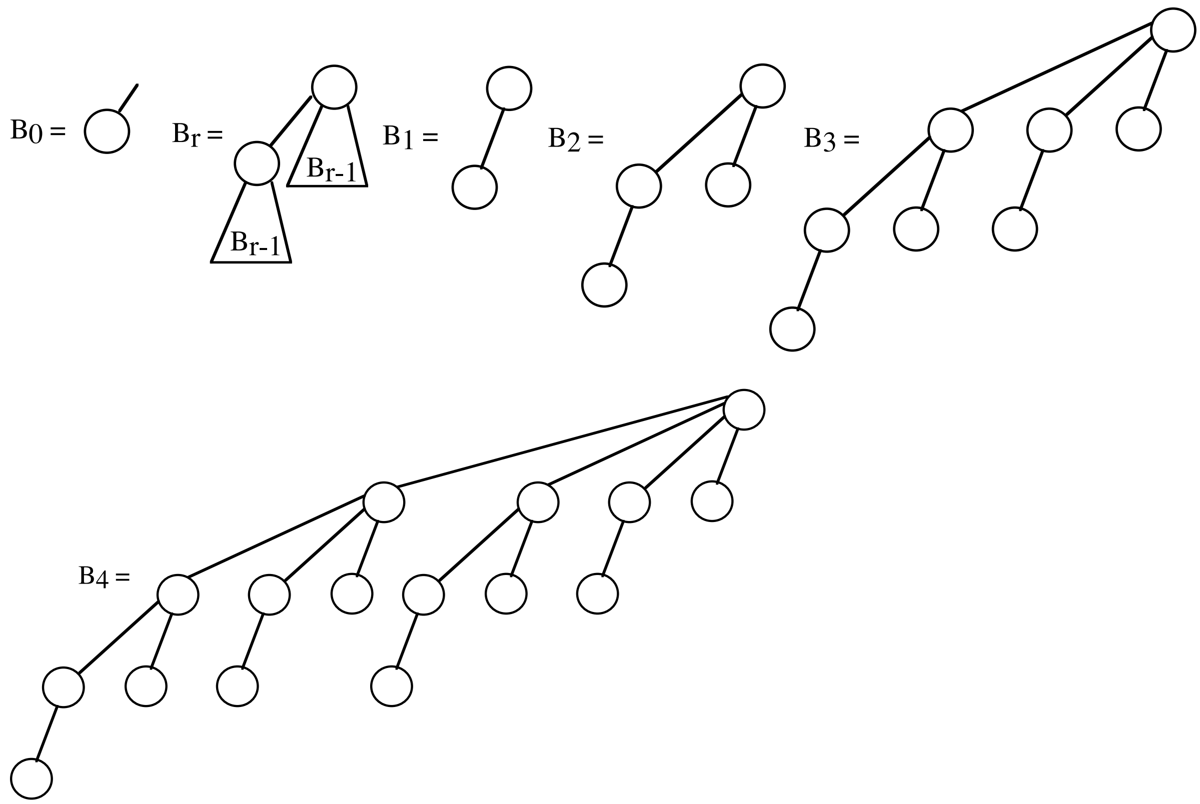


4. UNION X's subtree and modified original tree.

BINOMIAL HEAPS

Mergeable Heap - in O(log n) time

Based on Binomial Tree (with heap ordering) - $|B_r| = 2^r$



Binomial Heap = Forest of Binomial Trees

Each node includes priority, leftmost child, right sibling, parent, and degree.

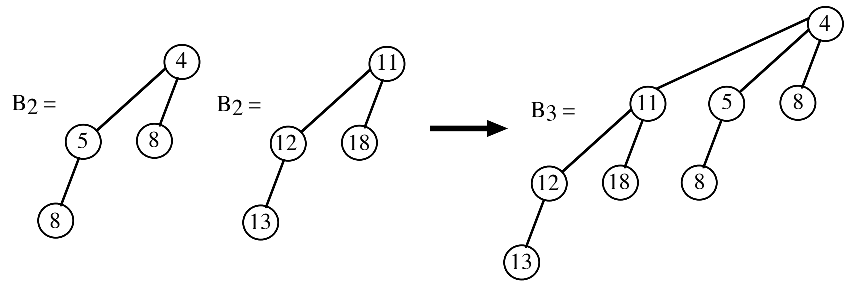
Tree roots are in a singly-linked list ordered by ascending degrees.

Children are in a singly-linked list ordered by descending degrees (could also use ascending).

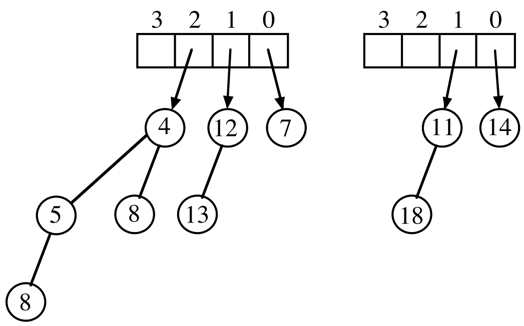
"Sack" idea can be used to reduce both space and time.

Can't have two B_i trees for any $i \Rightarrow$ Use binary representation of n.

Representation is useful for combining 2 B_i trees:



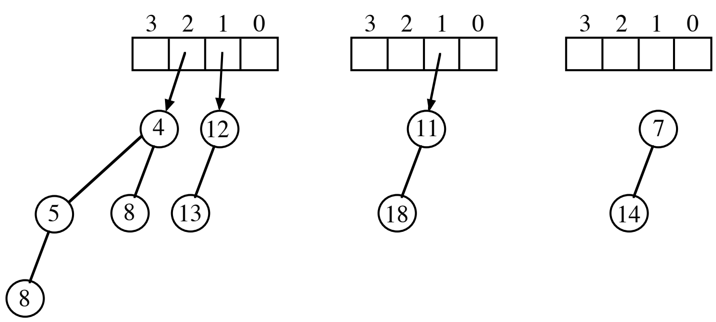
UNION of two binomial heaps



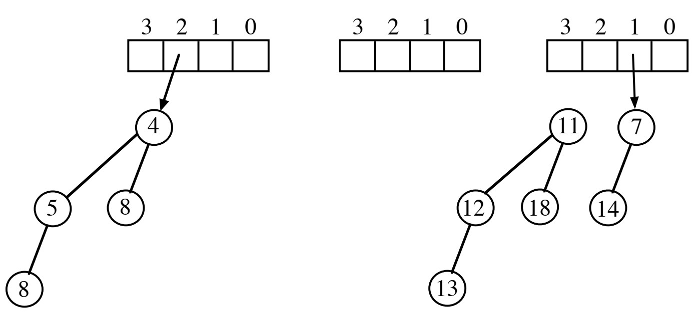
Based on binary addition:

$$0111 + 0011 = 1010$$

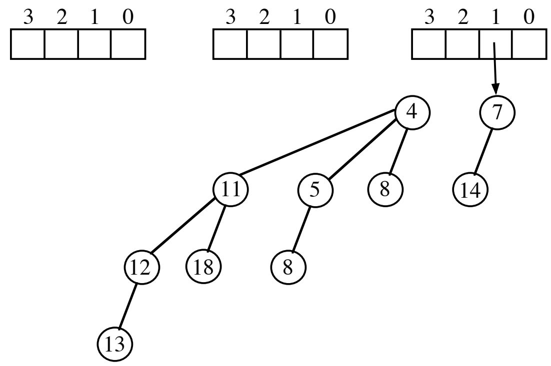
Link B_0 trees:



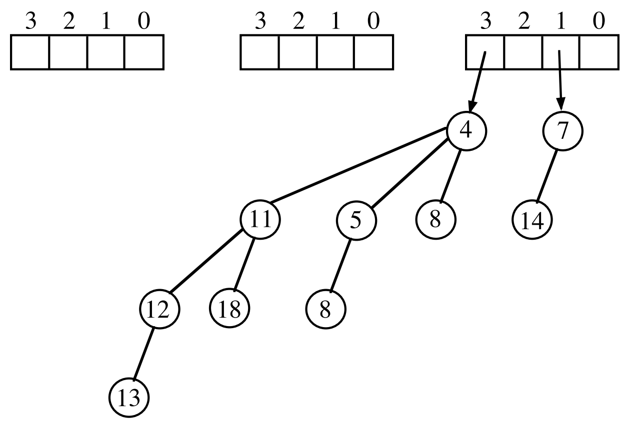
Link B_1 trees:



Link B_2 trees:



Save B_3 tree

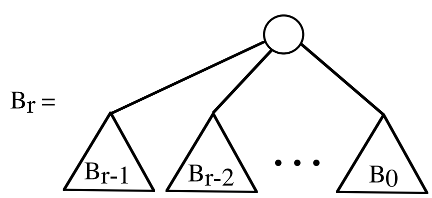


Insertion into binomial heap?

Implementing EXTRACT-MIN

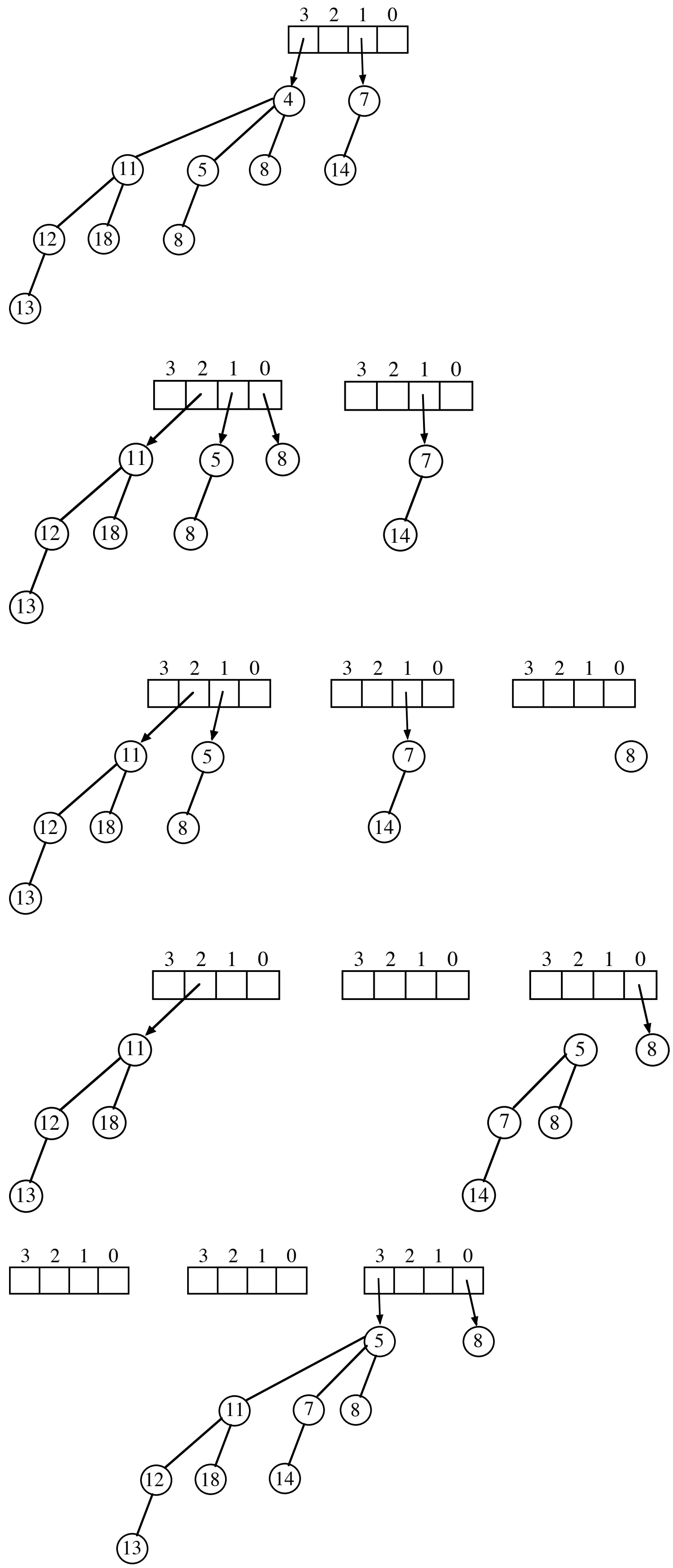
1. Scan tree roots for minimum key.

2. Decompose root of tree with minimum:



3. Treat fragments as binomial heap and UNION with remainder of original heap.

Example: Returns item 4 and decomposes the B_3 tree



Implementing DECREASE-KEY

Simply do exchanges through ancestor chain until min-heap property has been restored.

Suppose 13 is decreased to 6 in the previous example.

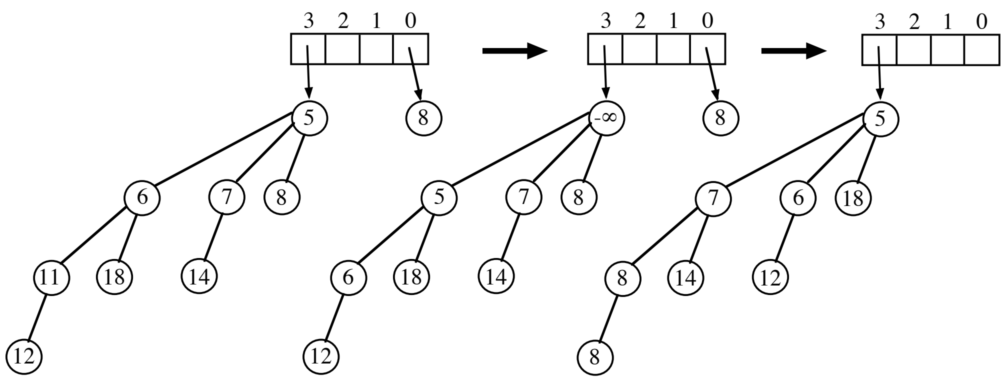
Implementing DELETE

1. Auxiliary data structure (see CSE 2320 Notes 5 regarding dictionary) is used to find the node to delete.

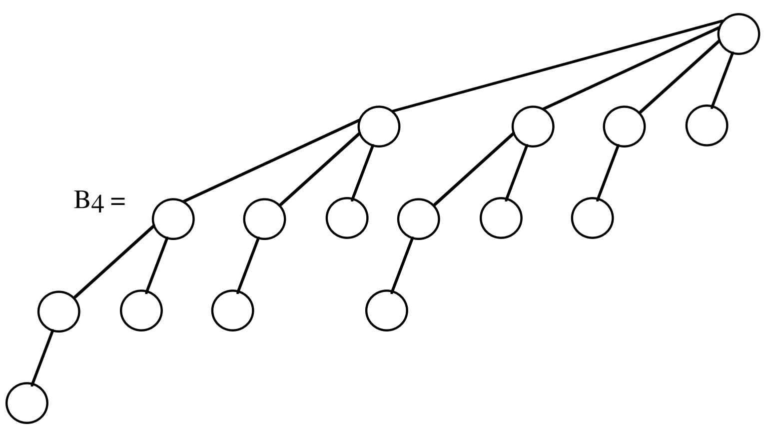
2. Use DECREASE-KEY to change priority to -∞.

3. Use EXTRACT-MIN to eliminate -∞.

Example: Delete 11.



Increase a key? What happens if obvious method is applied for key at root?



Binomial Heaps	vs.	Fibonacci Heaps
O(log n) actual costs		O(1) amortized, except EXTRACT-MIN and DELETE (O(log n) amortized)
		DECREASE-KEY is "faster"
Strict structural properties		Flexible structural properties (Allows laziness)
Analysis is straightforward		Amortized analysis involves subtle arguments regarding constants for asymptotic notation (especially for EXTRACT-MIN and cascading cut)

FIBONACCI HEAPS

Maintains pointer to root of tree with smallest priority.

If DELETE and DECREASE-KEY do not occur, then structure is like a binomial heap with ***multiple*** B_k trees. (Clean-up ("CONSOLIDATE") on EXTRACT-MIN and DELETE)

Otherwise:

1. A (k+1)-tree will be (initially) created from two k-trees, where k is number of children for root.

2. If a k-tree ***root*** loses a subtree, it is simply reclassified as a (k-1)-tree.

3. A ***non-root*** node x may lose one subtree and be "marked". If x loses another subtree, then x will be detached from its parent.

Observation:

1. x is any node in Fibonacci heap.

2. c_i is the ith child attached to x (not indicated in data structure).

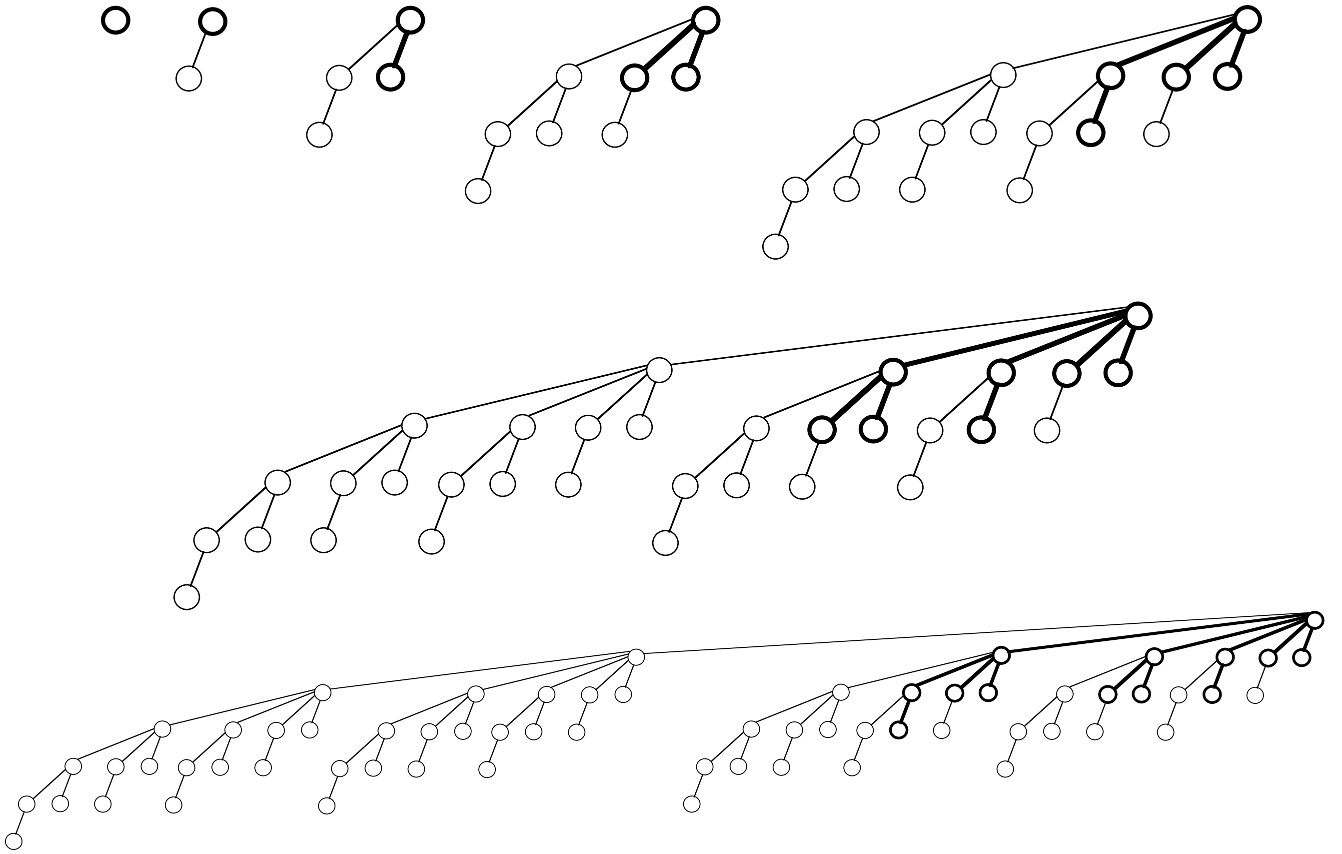
then c_i has at least i - 2 children.

Proof:

1. c_i had at least i - 1 children when attached to x.

2. It could have lost 1 child (assume it is for the largest subtree).

Minimum trees for each rank by pruning:

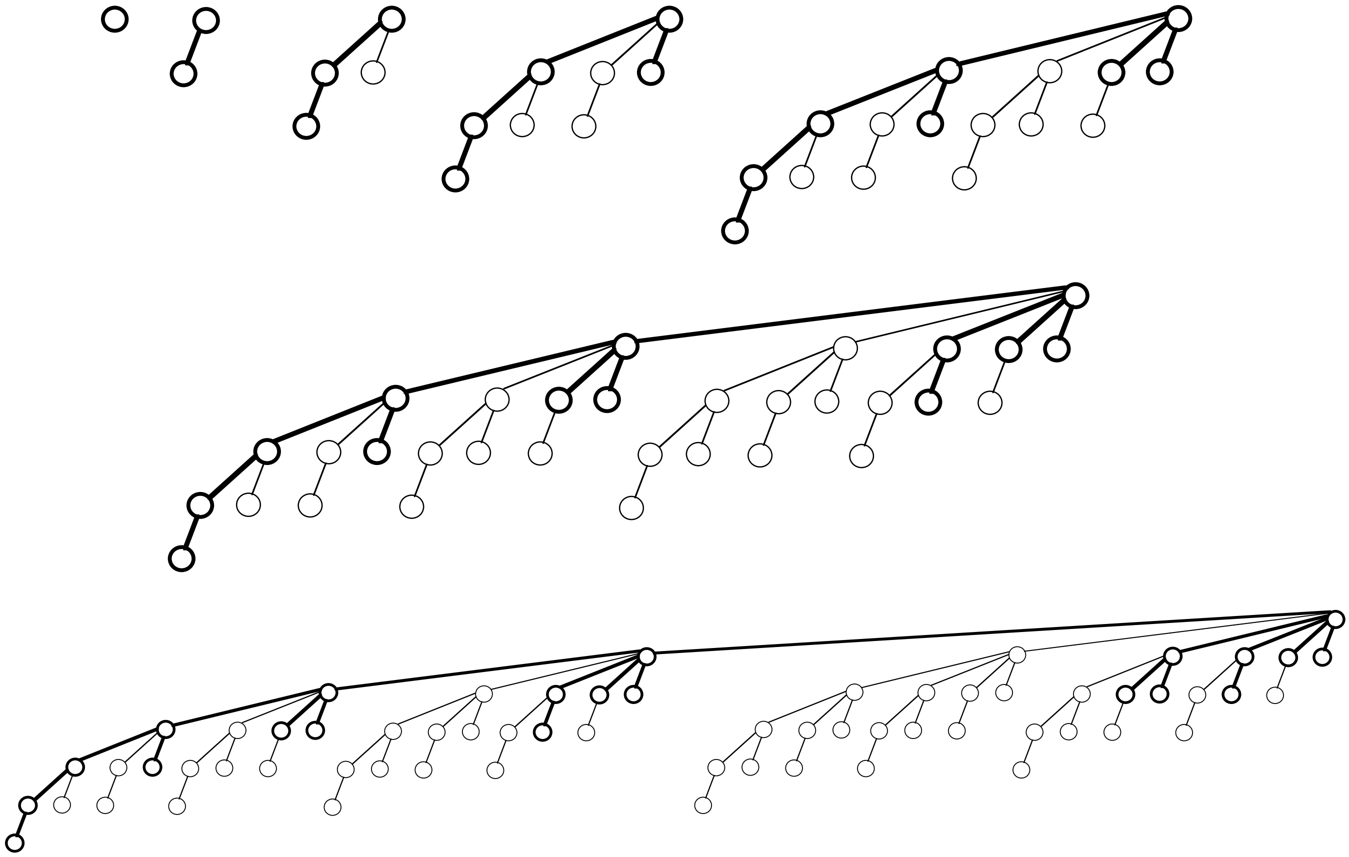


Most recently attached is always pruned

Original Rank	Pruned Rank	Height	Nodes
0	0	0	1
1	0	0	1
2	1	1	2
3	2	1	3
4	3	2	5
5	4	2	8
6	5	3	13
7	6	3	21
8	7	4	34
9	8	4	55
10	9	5	89
11	10	5	144
12	11	6	233
13	12	6	377
14	13	7	610
15	14	7	987
16	15	8	1597
17	16	8	2584
18	17	9	4181
19	18	9	6765
20	19	10	10946
21	20	10	17711
22	21	11	28657
23	22	11	46368
24	23	12	75025
25	24	12	121393
26	25	13	196418
27	26	13	317811
28	27	14	514229
29	28	14	832040
30	29	15	1346269
31	30	15	2178309

Ratio is 1.618034

Minimum trees for each height by modified pruning along longest path:

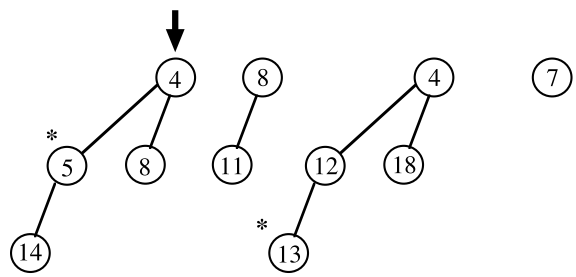


Second most recent is sometimes pruned to preserve longest path

Original Rank	Pruned Rank	Height	Nodes
0	0	0	1
1	0	1	2
2	1	2	3
3	2	3	5
4	3	4	8
5	4	5	13
6	5	6	21
7	6	7	34
8	7	8	55
9	8	9	89
10	9	10	144
11	10	11	233
12	11	12	377
13	12	13	610
14	13	14	987
15	14	15	1597
16	15	16	2584
17	16	17	4181
18	17	18	6765
19	18	19	10946
20	19	20	17711
21	20	21	28657
22	21	22	46368
23	22	23	75025
24	23	24	121393
25	24	25	196418
26	25	26	317811
27	26	27	514229
28	27	28	832040
29	28	29	1346269
30	29	30	2178309
31	30	31	3524578

Ratio is 1.618034

Simple Example



$\Phi(S) = p_t \bullet \#$ of trees $+ p_m \bullet \#$ of marks

Actual cost (c_i) for each operation is stated asymptotically. Implementation-dependent constants bound the actual cost of each operation and influence the values of p_t (traditionally valued 1) and p_m (traditionally valued 2).

UNION of two Fibonacci heaps

1. Append one list of trees to another.

2. Set pointer to new minimum key.

O(1) actual and amortized. (No change in Φ.)

INSERT

1. Create single node Fibonacci heap.

2. UNION

O(1) actual and amortized. (Φ goes up by p_t.)

EXTRACT-MIN

1. Remove minimum node (a root).

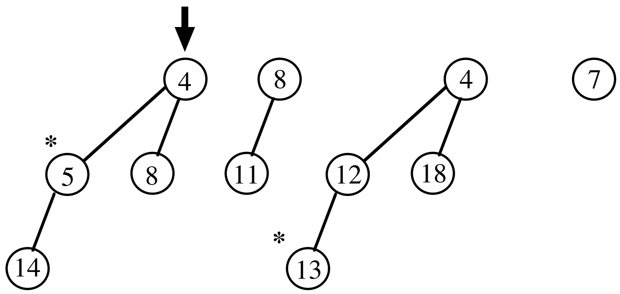
2. Append subtrees to list.

3. Much like binomial queue, use "accumulator" of pointers to CONSOLIDATE so that there is no more than one tree whose root has *k* children. (Root list is not ordered.) Initialization cost per accumulator entry is *d* (traditionally valued 1).

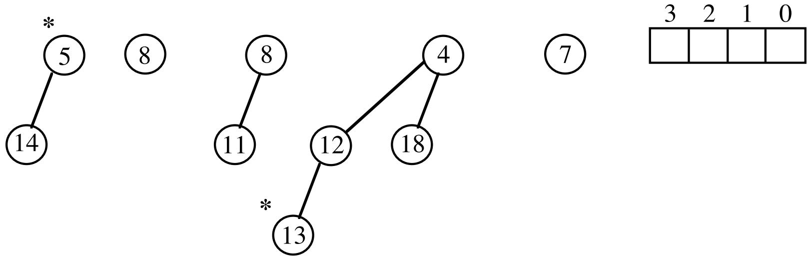
   Like binomial tree, two *k*-trees combine to give a (*k*+1)-tree. Combining cost is *e* (traditionally valued 1).

4. Must determine new minimum root.

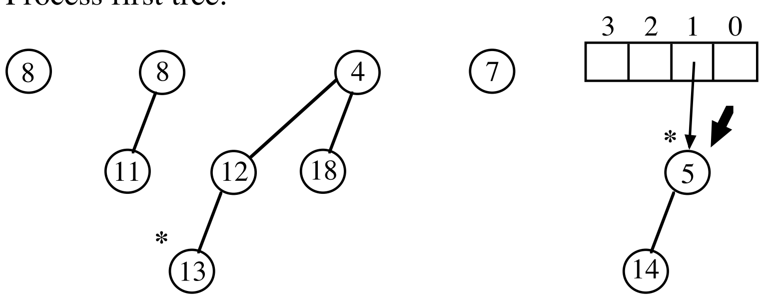
Actual cost is $d \bullet \log n + e \bullet \#$of trees = O(log *n* + # of trees). O(log *n*) amortized.



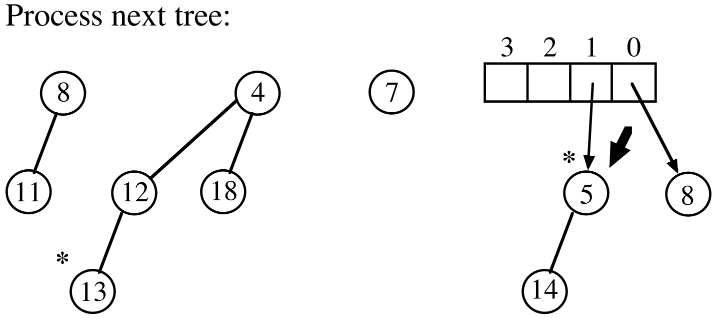
Remove minimum and decompose:



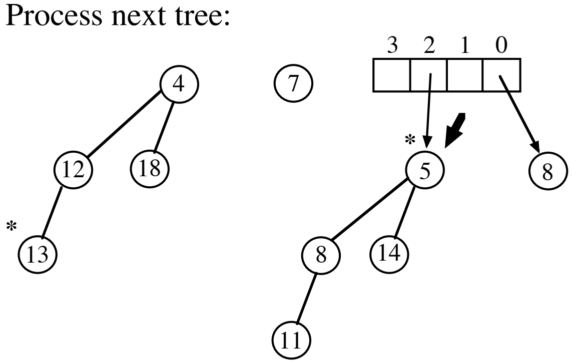
Process first tree:



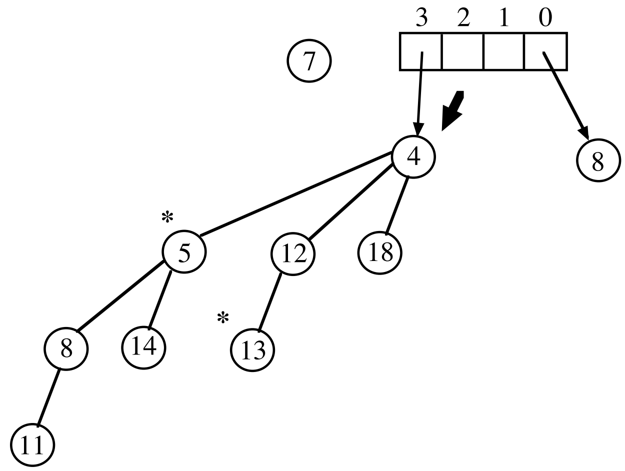
Process next tree:



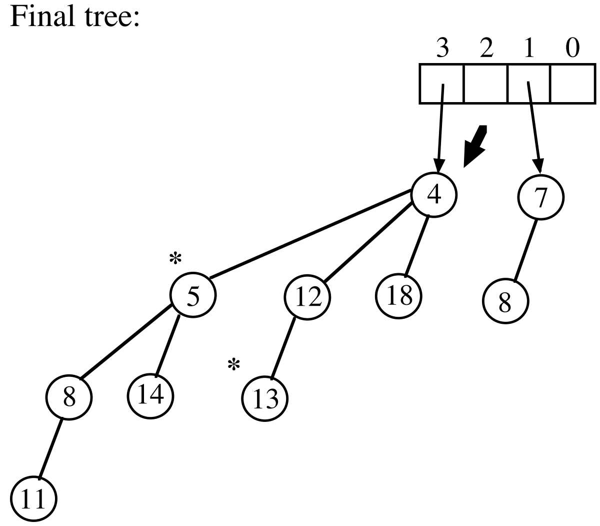
Process next tree:



Process next tree:



Final tree:



$\hat{C}_i = C_i + \Phi(S_i) - \Phi(S_{i-1})$

$C_i = d \bullet \log n + e \bullet \#$ of trees $= O(D(n) + \#$ of trees$)$

$D(n) =$ max # of children for any node in structure, including roots
(worst case is one tree in structure)

$\Phi(S_i) \le p_t(D(n)+1) + p_m \bullet \#$ of marked nodes
($D(n)+1$ is due to nodes with 0 ... $D(n)$ children)

$\Phi(S_{i-1}) = p_t \bullet \#$ of trees $+ p_m \bullet \#$ of marked nodes

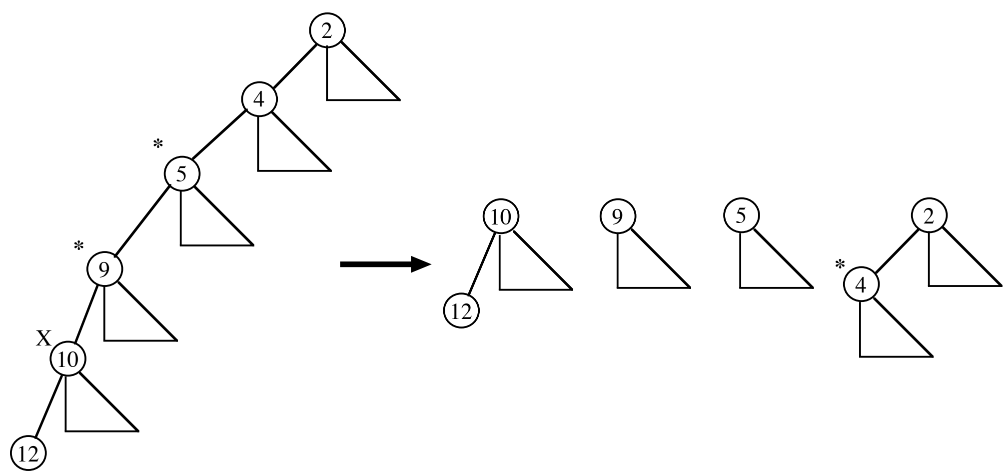
$\hat{C}_i = d \bullet \log n + e \bullet \#$ of trees $+ p_t(D(n)+1) - p_t \bullet \#$ of trees $= O(D(n))$ if $p_t \ge e$

Skim section 20.4, $D(n)$ is logarithmic in n

DECREASE-KEY and DELETE ⇒ Lose binomial heap properties

Based on *cascading cut* at X (that has a parent):

```
Clear mark on X                              // X is not necessarily marked
P := parent(X)
Break (cut) link from X to P
X := P
P := parent(X)
while P ≠ nil
        if X is marked
                Break (cut) link from X to P
                Clear mark on X              // X definitely is marked
                X := P
                P := parent(X)
        else
                Set mark on X                // X cannot be the root
                P := nil
```



DECREASE-KEY

Not based on swaps (as done for binary and binomial heaps), but similar to leftist heaps

1. Decrease key value.

2. If node has parent and key < parent's key
   Perform cascading cut at key's node

3. Check if key is lowest in structure

Actual cost is O(log *n*) based on the number of cuts. O(1) amortized

DELETE

1. DECREASE-KEY value to $-\infty$.

2. EXTRACT-MIN

EXTRACT-MIN dominates actual cost. $O(\log n)$ amortized.

Amortized Cost of Cascading Cut

Suppose c is the number of cuts.

$c_i = fc = O(c)$ $\quad f$ is the actual cost of cutting a node (traditionally valued 1)

$\Phi(S_i) \le p_t \bullet (\#\ \text{trees} + c) + p_m(\#\,\text{marked nodes} - c + 2)$

$\Phi(S_{i-1}) = p_t \bullet \#\ \text{trees} + p_m \bullet \#\,\text{marked nodes}$

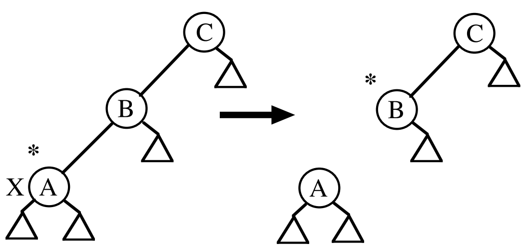
$\hat{c}_i = fc + p_t c + p_m(-c + 2) = c(f + p_t - p_m) + 2p_m = O(1)$ if $p_m \ge f + p_t$

The $-c + 2$ upper bound on the change in marked nodes is based on the following observations about the c cuts that occur:

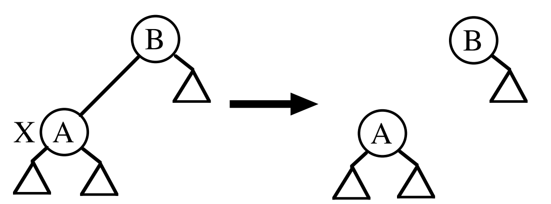
1. The first cut loses a mark only when the initial node X is marked.

2. Cuts 2 through c - 1 must lose a mark.

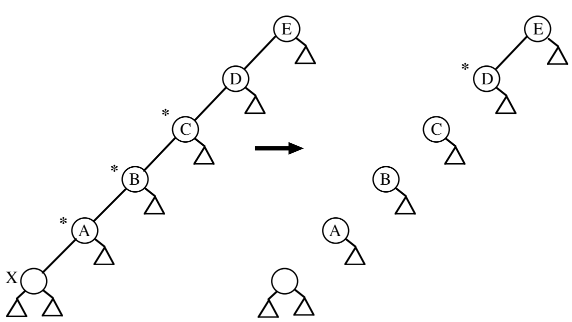
3. The last cut loses a mark only when the parent node is the root.



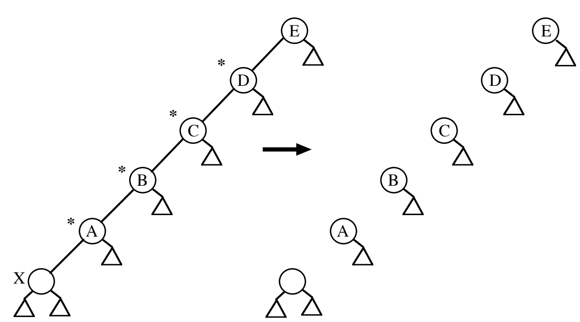
c = 1, change in marked nodes is $-c + 2 \ge 0$



c = 1, change in marked nodes is $-c + 2 \ge 0$



c = 4, change in marked nodes is $-c + 2 = -2$



c = 5, change in marked nodes is $-c + 2 \ge -4$