

## CSE 5311 Notes 7b: van Emde Boas Trees

(Last updated 6/22/13 1:36 PM)

CLRS 20.3 (20.2 on proto vEB is nice, but . . .)

Operations supported in  $O(\log \log u)$  time for keys in domain  $0 \dots u - 1$ ,  $u = 2^k$ :

SEARCH (MEMBER)

INSERT

DELETE

MINIMUM -  $\Theta(1)$

MAXIMUM -  $\Theta(1)$

SUCCESSOR

PREDECESSOR

A simple implementation will use  $\Theta(u)$  initialization time and space, but these can be reduced to  $\Theta(1)$  and  $\Theta(n)$  by using NULL pointers for empty portions of tree and replacing bit vectors and pointer tables with hash tables. (See CLRS problem 20-1.)

|          | Subtree Domain<br>Size | # of Children<br>Subtree Root  | Domain Size for<br>Child Subtrees |
|----------|------------------------|--|-----------------------------------|
|          | $2^{2k}$               | $2^k$  | $2^k$                             |
|          | $2^{2k+1}$             | $2^{k+1}$  | $2^k$                             |
| Example: | $2^{22}$               | $2^{11}$   | $2^{11}$                          |
|          | $2^{11}$               | $2^6$  | $2^5$                             |
|          | $2^5$                  | $2^3$  | $2^2$                             |
|          | $2^2$                  | $2^1$  | $2^1$                             |
|          | $2^1$ (leaves)         |  |                                   |
|          |                        | ↑  |                                   |
|          |                        | High-order bits for SEARCH in $\Theta(\log \log u)$<br>(Can also line-up leaves contiguously for $\Theta(1)$ ) |                                   |

Also, see high, low, and index functions in CLRS, p. 546.

Each node includes the *min* and *max* used within the subtree, to give  $\Theta(1)$  MINIMUM and MAXIMUM, but these also help (timewise) with SUCCESSOR and PREDECESSOR.

If a subtree has no members, then  $min = max = NIL (/)$ .

If a subtree has one member, then  $min = max$  and no members are stored below the subtree root.

If a subtree has exactly two members, then  $min < max$ , and only members  $> min$  are stored below the subtree root. (To repeat, *min* is not stored in any of the clusters below.)

Even with *min* and *max*, some SUCCESSOR/ PREDECESSOR requests lead to scans of children, so summary structures are added. These are recursive, so there are summaries within summaries.

Concept: When a tree's domain size  $2^{2k}$  (or  $2^{2k+1}$ ) is larger than  $2^1$ , there is a *summary structure* of domain size  $2^k$  (or  $2^{k+1}$ ) that will have *i* as a member if and only if sub-cluster *i* of the tree has at least one member.

(CLRS, p. 548 - Figure 20.6, members are {2, 3, 4, 5, 7, 14, 15})

```

root (base 0) u 16 min 2 (2) max 15
  summary (base 0) u 4 min 0 (0) max 3
    summary (base 0) u 2 min 0 (0) max 1 (1)
      cluster[0] (base 0) u 2 min 1 (1) max 1
      cluster[1] (base 2) u 2 min 1 (3) max 1
    cluster[0] (base 0) u 4 min 3 (3) max 3
      summary (base 0) u 2 min / max /
      cluster[0] (base 0) u 2 min / max /
      cluster[1] (base 2) u 2 min / max /
    cluster[1] (base 4) u 4 min 0 (4) max 3
      summary (base 0) u 2 min 0 (0) max 1 (1)
      cluster[0] (base 4) u 2 min 1 (5) max 1
      cluster[1] (base 6) u 2 min 1 (7) max 1
    cluster[2] (base 8) u 4 min / max /
      summary (base 0) u 2 min / max /
      cluster[0] (base 8) u 2 min / max /
      cluster[1] (base 10) u 2 min / max /
    cluster[3] (base 12) u 4 min 2 (14) max 3
      summary (base 0) u 2 min 1 (1) max 1
      cluster[0] (base 12) u 2 min / max /
      cluster[1] (base 14) u 2 min 1 (15) max 1

```

Later diagrams will not show empty, untouched clusters . . .

Diagrams were produced using `real.book.c` and `real.driver.c` on course webpage.

A larger example: {2, 4, 7, 10, 11, 12, 13, 14, 15, 20, 26, 45, 46, 47, 48, 49, 50, 52, 54, 56, 57, 58, 59}

```

root (base 0) u 64 min 2 (2) max 59 [1: minimum=2]
  summary (base 0) u 8 min 0 (0) max 7
    summary (base 0) u 4 min 0 (0) max 3
      summary (base 0) u 2 min 0 (0) max 1 (1)
        cluster[0] (base 0) u 2 min 1 (1) max 1
        cluster[1] (base 2) u 2 min 0 (2) max 1 (3)
      cluster[0] (base 0) u 2 min 1 (1) max 1
      cluster[1] (base 2) u 2 min 0 (2) max 1 (3)
      cluster[2] (base 4) u 2 min 1 (5) max 1
      cluster[3] (base 6) u 2 min 0 (6) max 1 (7)
    cluster[0] (base 0) u 8 min 4 (4) max 7
      summary (base 0) u 4 min 3 (3) max 3
      cluster[3] (base 6) u 2 min 1 (7) max 1
    cluster[1] (base 8) u 8 min 2 (10) max 7
      summary (base 0) u 4 min 1 (1) max 3
        summary (base 0) u 2 min 1 (1) max 1
          cluster[1] (base 2) u 2 min 0 (2) max 1 (3)
        cluster[1] (base 10) u 2 min 1 (11) max 1
        cluster[2] (base 12) u 2 min 0 (12) max 1 (13)
        cluster[3] (base 14) u 2 min 0 (14) max 1 (15)
    cluster[2] (base 16) u 8 min 4 (20) max 4
    cluster[3] (base 24) u 8 min 2 (26) max 2
    cluster[5] (base 40) u 8 min 5 (45) max 7
      summary (base 0) u 4 min 3 (3) max 3
      cluster[3] (base 46) u 2 min 0 (46) max 1 (47)
    cluster[6] (base 48) u 8 min 0 (48) max 6
      summary (base 0) u 4 min 0 (0) max 3
        summary (base 0) u 2 min 0 (0) max 1 (1)
          cluster[0] (base 0) u 2 min 1 (1) max 1
          cluster[1] (base 2) u 2 min 0 (2) max 1 (3)
        cluster[0] (base 48) u 2 min 1 (49) max 1
        cluster[1] (base 50) u 2 min 0 (50) max 0
        cluster[2] (base 52) u 2 min 0 (52) max 0
        cluster[3] (base 54) u 2 min 0 (54) max 0
    cluster[7] (base 56) u 8 min 0 (56) max 3
      summary (base 0) u 4 min 0 (0) max 1
        summary (base 0) u 2 min 0 (0) max 0
          cluster[0] (base 0) u 2 min 1 (1) max 1
        cluster[0] (base 56) u 2 min 1 (57) max 1
        cluster[1] (base 58) u 2 min 0 (58) max 1 (59)

```

Also:

```

root (base 0) u 64 min 2 (2) max 59 [1: maximum=59]
.
.
.

```

50 (110010) is a member by touching three nodes:

```

root (base 0) u 64 min 2 (2) max 59 [1: member(50)=member(cluster[6],2)]
  summary (base 0) u 8 min 0 (0) max 7
    summary (base 0) u 4 min 0 (0) max 3
      summary (base 0) u 2 min 0 (0) max 1 (1)
        cluster[0] (base 0) u 2 min 1 (1) max 1
          cluster[1] (base 2) u 2 min 0 (2) max 1 (3)
            cluster[0] (base 0) u 2 min 1 (1) max 1
              cluster[1] (base 2) u 2 min 0 (2) max 1 (3)
                cluster[2] (base 4) u 2 min 1 (5) max 1
                  cluster[3] (base 6) u 2 min 0 (6) max 1 (7)
                    cluster[0] (base 0) u 8 min 4 (4) max 7
                      summary (base 0) u 4 min 3 (3) max 3
                        cluster[3] (base 6) u 2 min 1 (7) max 1
                          cluster[1] (base 8) u 8 min 2 (10) max 7
                            summary (base 0) u 4 min 1 (1) max 3
                              summary (base 0) u 2 min 1 (1) max 1
                                cluster[1] (base 2) u 2 min 0 (2) max 1 (3)
                                  cluster[1] (base 10) u 2 min 1 (11) max 1
                                    cluster[2] (base 12) u 2 min 0 (12) max 1 (13)
                                      cluster[3] (base 14) u 2 min 0 (14) max 1 (15)
                                        cluster[2] (base 16) u 8 min 4 (20) max 4
                                          cluster[3] (base 24) u 8 min 2 (26) max 2
                                            cluster[5] (base 40) u 8 min 5 (45) max 7
                                              summary (base 0) u 4 min 3 (3) max 3
                                                cluster[3] (base 46) u 2 min 0 (46) max 1 (47)
                                                  cluster[6] (base 48) u 8 min 0 (48) max 6 [2: member(2)=member(cluster[1],0)]
                                                    summary (base 0) u 4 min 0 (0) max 3
                                                      summary (base 0) u 2 min 0 (0) max 1 (1)
                                                        cluster[0] (base 0) u 2 min 1 (1) max 1
                                                          cluster[1] (base 2) u 2 min 0 (2) max 1 (3)
                                                            cluster[0] (base 48) u 2 min 1 (49) max 1
                                                              cluster[1] (base 50) u 2 min 0 (50) max 0 [3: member(0)=1]
                                                                cluster[2] (base 52) u 2 min 0 (52) max 0
                                                                  cluster[3] (base 54) u 2 min 0 (54) max 0
                                                                    cluster[7] (base 56) u 8 min 0 (56) max 3
                                                                      summary (base 0) u 4 min 0 (0) max 1
                                                                        summary (base 0) u 2 min 0 (0) max 0
                                                                          cluster[0] (base 0) u 2 min 1 (1) max 1
                                                                            cluster[0] (base 56) u 2 min 1 (57) max 1
                                                                              cluster[1] (base 58) u 2 min 0 (58) max 1 (59)

```

Successor of 0 is 2:

```

root (base 0) u 64 min 2 (2) max 59 [1: root successor(0)=V->min=2]

```

Successor of 49 (110001) is 50:

```

root (base 0) u 64 min 2 (2) max 59 [1: successor(49)
                                     2: must descend
                                     4: maxLow=6
                                     20: descent successor(49)=50]
summary (base 0) u 8 min 0 (0) max 7
  summary (base 0) u 4 min 0 (0) max 3
    summary (base 0) u 2 min 0 (0) max 1 (1)
    cluster[0] (base 0) u 2 min 1 (1) max 1
    cluster[1] (base 2) u 2 min 0 (2) max 1 (3)
  cluster[0] (base 0) u 2 min 1 (1) max 1
  cluster[1] (base 2) u 2 min 0 (2) max 1 (3)
  cluster[2] (base 4) u 2 min 1 (5) max 1
  cluster[3] (base 6) u 2 min 0 (6) max 1 (7)
cluster[0] (base 0) u 8 min 4 (4) max 7
  summary (base 0) u 4 min 3 (3) max 3
  cluster[3] (base 6) u 2 min 1 (7) max 1
cluster[1] (base 8) u 8 min 2 (10) max 7
  summary (base 0) u 4 min 1 (1) max 3
    summary (base 0) u 2 min 1 (1) max 1
    cluster[1] (base 2) u 2 min 0 (2) max 1 (3)
  cluster[1] (base 10) u 2 min 1 (11) max 1
  cluster[2] (base 12) u 2 min 0 (12) max 1 (13)
  cluster[3] (base 14) u 2 min 0 (14) max 1 (15)
cluster[2] (base 16) u 8 min 4 (20) max 4
cluster[3] (base 24) u 8 min 2 (26) max 2
cluster[5] (base 40) u 8 min 5 (45) max 7
  summary (base 0) u 4 min 3 (3) max 3
  cluster[3] (base 46) u 2 min 0 (46) max 1 (47)
cluster[6] (base 48) u 8 min 0 (48) max 6 [3: maximum=6
                                         5: successor(1)
                                         6: must descend
                                         8: maxLow=1
                                         9: descent failed
                                         19: neighbor successor(1)=2]
  summary (base 0) u 4 min 0 (0) max 3 [10: successor(0)
                                         11: must descend
                                         13: maxLow=1
                                         16: descent successor(0)=1]
    summary (base 0) u 2 min 0 (0) max 1 (1)
    cluster[0] (base 0) u 2 min 1 (1) max 1 [12: maximum=1
                                             14: successor(0)
                                             15: leaf successor(0)=1]
      cluster[1] (base 2) u 2 min 0 (2) max 1 (3)
    cluster[0] (base 48) u 2 min 1 (49) max 1 [7: maximum=1]
    cluster[1] (base 50) u 2 min 0 (50) max 0 [17: use succCluster
                                             18: minimum=0]
      cluster[2] (base 52) u 2 min 0 (52) max 0
      cluster[3] (base 54) u 2 min 0 (54) max 0
cluster[7] (base 56) u 8 min 0 (56) max 3
  summary (base 0) u 4 min 0 (0) max 1
    summary (base 0) u 2 min 0 (0) max 0
    cluster[0] (base 0) u 2 min 1 (1) max 1
  cluster[0] (base 56) u 2 min 1 (57) max 1
  cluster[1] (base 58) u 2 min 0 (58) max 1 (59)

```

Successor of 26 (011010) is 45:

```

root (base 0) u 64 min 2 (2) max 59 [1: successor(26)
    2: must descend
    4: maxLow=2
    5: descent failed
    26: neighbor successor(26)=45]
summary (base 0) u 8 min 0 (0) max 7 [6: successor(3)
    7: must descend
    9: maxLow=1
    10: descent failed
    23: neighbor successor(3)=5]
summary (base 0) u 4 min 0 (0) max 3 [11: successor(1)
    12: must descend
    14: maxLow=1
    15: descent failed
    20: neighbor successor(1)=2]
summary (base 0) u 2 min 0 (0) max 1 (1) [16: successor(0)
    17: leaf successor(0)=1]
cluster[0] (base 0) u 2 min 1 (1) max 1 [13: maximum=1]
cluster[1] (base 2) u 2 min 0 (2) max 1 (3) [18: use succCluster
    19: minimum=0]

cluster[0] (base 0) u 2 min 1 (1) max 1
cluster[1] (base 2) u 2 min 0 (2) max 1 (3) [8: maximum=1]
cluster[2] (base 4) u 2 min 1 (5) max 1 [21: use succCluster
    22: minimum=1]

cluster[3] (base 6) u 2 min 0 (6) max 1 (7)
cluster[0] (base 0) u 8 min 4 (4) max 7
summary (base 0) u 4 min 3 (3) max 3
cluster[3] (base 6) u 2 min 1 (7) max 1
cluster[1] (base 8) u 8 min 2 (10) max 7
summary (base 0) u 4 min 1 (1) max 3
summary (base 0) u 2 min 1 (1) max 1
cluster[1] (base 2) u 2 min 0 (2) max 1 (3)
cluster[1] (base 10) u 2 min 1 (11) max 1
cluster[2] (base 12) u 2 min 0 (12) max 1 (13)
cluster[3] (base 14) u 2 min 0 (14) max 1 (15)
cluster[2] (base 16) u 8 min 4 (20) max 4
cluster[3] (base 24) u 8 min 2 (26) max 2 [3: maximum=2]
cluster[5] (base 40) u 8 min 5 (45) max 7 [24: use succCluster
    25: minimum=5]

summary (base 0) u 4 min 3 (3) max 3
cluster[3] (base 46) u 2 min 0 (46) max 1 (47)
cluster[6] (base 48) u 8 min 0 (48) max 6
summary (base 0) u 4 min 0 (0) max 3
summary (base 0) u 2 min 0 (0) max 1 (1)
cluster[0] (base 0) u 2 min 1 (1) max 1
cluster[1] (base 2) u 2 min 0 (2) max 1 (3)
cluster[0] (base 48) u 2 min 1 (49) max 1
cluster[1] (base 50) u 2 min 0 (50) max 0
cluster[2] (base 52) u 2 min 0 (52) max 0
cluster[3] (base 54) u 2 min 0 (54) max 0
cluster[7] (base 56) u 8 min 0 (56) max 3
summary (base 0) u 4 min 0 (0) max 1
summary (base 0) u 2 min 0 (0) max 0
cluster[0] (base 0) u 2 min 1 (1) max 1
cluster[0] (base 56) u 2 min 1 (57) max 1
cluster[1] (base 58) u 2 min 0 (58) max 1 (59)

```

Predecessor of 58 (111010) is 57:

```

root (base 0) u 64 min 2 (2) max 59 [1: predecessor(58)
    2: must descend
    4: minLow=0
    21: descent predecessor(58)=57]
summary (base 0) u 8 min 0 (0) max 7
  summary (base 0) u 4 min 0 (0) max 3
    summary (base 0) u 2 min 0 (0) max 1 (1)
      cluster[0] (base 0) u 2 min 1 (1) max 1
      cluster[1] (base 2) u 2 min 0 (2) max 1 (3)
    cluster[0] (base 0) u 2 min 1 (1) max 1
    cluster[1] (base 2) u 2 min 0 (2) max 1 (3)
    cluster[2] (base 4) u 2 min 1 (5) max 1
    cluster[3] (base 6) u 2 min 0 (6) max 1 (7)
  cluster[0] (base 0) u 8 min 4 (4) max 7
    summary (base 0) u 4 min 3 (3) max 3
      cluster[3] (base 6) u 2 min 1 (7) max 1
  cluster[1] (base 8) u 8 min 2 (10) max 7
    summary (base 0) u 4 min 1 (1) max 3
      summary (base 0) u 2 min 1 (1) max 1
        cluster[1] (base 2) u 2 min 0 (2) max 1 (3)
      cluster[1] (base 10) u 2 min 1 (11) max 1
      cluster[2] (base 12) u 2 min 0 (12) max 1 (13)
      cluster[3] (base 14) u 2 min 0 (14) max 1 (15)
  cluster[2] (base 16) u 8 min 4 (20) max 4
  cluster[3] (base 24) u 8 min 2 (26) max 2
  cluster[5] (base 40) u 8 min 5 (45) max 7
    summary (base 0) u 4 min 3 (3) max 3
      cluster[3] (base 46) u 2 min 0 (46) max 1 (47)
  cluster[6] (base 48) u 8 min 0 (48) max 6
    summary (base 0) u 4 min 0 (0) max 3
      summary (base 0) u 2 min 0 (0) max 1 (1)
        cluster[0] (base 0) u 2 min 1 (1) max 1
        cluster[1] (base 2) u 2 min 0 (2) max 1 (3)
      cluster[0] (base 48) u 2 min 1 (49) max 1
      cluster[1] (base 50) u 2 min 0 (50) max 0
      cluster[2] (base 52) u 2 min 0 (52) max 0
      cluster[3] (base 54) u 2 min 0 (54) max 0
  cluster[7] (base 56) u 8 min 0 (56) max 3 [3: minimum=0
    5: predecessor(2)
    6: must descend
    8: minLow=0
    9: descent failed
    20: neighbor predecessor(2)=1]
  summary (base 0) u 4 min 0 (0) max 1 [10: predecessor(1)
    11: must descend
    13: minLow=1
    14: descent failed
    17: special - predecessor=V->min=0]
    summary (base 0) u 2 min 0 (0) max 0 [15: predecessor(0)
      16: no leaf predecessor(0)
        =vEBNIL]
      cluster[0] (base 0) u 2 min 1 (1) max 1 [12: minimum=1]
    cluster[0] (base 56) u 2 min 1 (57) max 1 [18: use predCluster
      19: maximum=1]
    cluster[1] (base 58) u 2 min 0 (58) max 1 (59) [7: minimum=0]

```

Predecessor of 49 (110001) is 48:

```

root (base 0) u 64 min 2 (2) max 59 [1: predecessor(49)
      2: must descend
      4: minLow=0
      19: descent predecessor(49)=48]
summary (base 0) u 8 min 0 (0) max 7
  summary (base 0) u 4 min 0 (0) max 3
    summary (base 0) u 2 min 0 (0) max 1 (1)
      cluster[0] (base 0) u 2 min 1 (1) max 1
      cluster[1] (base 2) u 2 min 0 (2) max 1 (3)
    cluster[0] (base 0) u 2 min 1 (1) max 1
    cluster[1] (base 2) u 2 min 0 (2) max 1 (3)
    cluster[2] (base 4) u 2 min 1 (5) max 1
    cluster[3] (base 6) u 2 min 0 (6) max 1 (7)
  cluster[0] (base 0) u 8 min 4 (4) max 7
    summary (base 0) u 4 min 3 (3) max 3
      cluster[3] (base 6) u 2 min 1 (7) max 1
  cluster[1] (base 8) u 8 min 2 (10) max 7
    summary (base 0) u 4 min 1 (1) max 3
      summary (base 0) u 2 min 1 (1) max 1
        cluster[1] (base 2) u 2 min 0 (2) max 1 (3)
      cluster[1] (base 10) u 2 min 1 (11) max 1
      cluster[2] (base 12) u 2 min 0 (12) max 1 (13)
      cluster[3] (base 14) u 2 min 0 (14) max 1 (15)
    cluster[2] (base 16) u 8 min 4 (20) max 4
    cluster[3] (base 24) u 8 min 2 (26) max 2
    cluster[5] (base 40) u 8 min 5 (45) max 7
      summary (base 0) u 4 min 3 (3) max 3
        cluster[3] (base 46) u 2 min 0 (46) max 1 (47)
    cluster[6] (base 48) u 8 min 0 (48) max 6 [3: minimum=0
      5: predecessor(1)
      6: must descend
      8: minLow=1
      9: descent failed
      18: special - predecessor=V->min=0]
    summary (base 0) u 4 min 0 (0) max 3 [10: predecessor(0)
      11: must descend
      13: minLow=1
      14: descent failed
      17: no neighbor predecessor(0)
      =vEBNIL]
      summary (base 0) u 2 min 0 (0) max 1 (1) [15: predecessor(0)
        16: no leaf predecessor(0)
        =vEBNIL]
        cluster[0] (base 0) u 2 min 1 (1) max 1 [12: minimum=1]
        cluster[1] (base 2) u 2 min 0 (2) max 1 (3)
      cluster[0] (base 48) u 2 min 1 (49) max 1 [7: minimum=1]
      cluster[1] (base 50) u 2 min 0 (50) max 0
      cluster[2] (base 52) u 2 min 0 (52) max 0
      cluster[3] (base 54) u 2 min 0 (54) max 0
    cluster[7] (base 56) u 8 min 0 (56) max 3
      summary (base 0) u 4 min 0 (0) max 1
        summary (base 0) u 2 min 0 (0) max 0
          cluster[0] (base 0) u 2 min 1 (1) max 1
        cluster[0] (base 56) u 2 min 1 (57) max 1
        cluster[1] (base 58) u 2 min 0 (58) max 1 (59)

```

Insert of 42 (101010) completed:

```

root (base 0) u 64 min 2 (2) max 59 [1: insert(42)]
  summary (base 0) u 8 min 0 (0) max 7
    summary (base 0) u 4 min 0 (0) max 3
      summary (base 0) u 2 min 0 (0) max 1 (1)
        cluster[0] (base 0) u 2 min 1 (1) max 1
          cluster[1] (base 2) u 2 min 0 (2) max 1 (3)
            cluster[0] (base 0) u 2 min 1 (1) max 1
              cluster[1] (base 2) u 2 min 0 (2) max 1 (3)
                cluster[2] (base 4) u 2 min 1 (5) max 1
                  cluster[3] (base 6) u 2 min 0 (6) max 1 (7)
                    cluster[0] (base 0) u 8 min 4 (4) max 7
                      summary (base 0) u 4 min 3 (3) max 3
                        cluster[3] (base 6) u 2 min 1 (7) max 1
                          cluster[1] (base 8) u 8 min 2 (10) max 7
                            summary (base 0) u 4 min 1 (1) max 3
                              summary (base 0) u 2 min 1 (1) max 1
                                cluster[1] (base 2) u 2 min 0 (2) max 1 (3)
                                  cluster[1] (base 10) u 2 min 1 (11) max 1
                                    cluster[2] (base 12) u 2 min 0 (12) max 1 (13)
                                      cluster[3] (base 14) u 2 min 0 (14) max 1 (15)
                                        cluster[2] (base 16) u 8 min 4 (20) max 4
                                          cluster[3] (base 24) u 8 min 2 (26) max 2
                                            cluster[5] (base 40) u 8 min 2 (42) max 7 [2: minimum=5
                                                                 3: insert into non-empty sub-cluster
                                                                 4: insert(2)
                                                                 5: swapped arg 2 with V->min 5]
                                              summary (base 0) u 4 min 2 (2) max 3 [8: insert(2)
                                                                 9: swapped arg 2 with V->min 3]
                                                summary (base 0) u 2 min 1 (1) max 1 [12: insert(1)
                                                                 13: emptyTreeInsert(1)]
                                                  cluster[1] (base 2) u 2 min 1 (3) max 1 [10: minimum=-1
                                                                 11: insert into empty sub-
                                                                 cluster
                                                                 14: emptyTreeInsert(1)]
                                                    cluster[2] (base 44) u 2 min 1 (45) max 1 [6: minimum=-1
                                                                 7: insert into empty sub-cluster
                                                                 15: emptyTreeInsert(1)]
                                                      cluster[3] (base 46) u 2 min 0 (46) max 1 (47)
                                                        cluster[6] (base 48) u 8 min 0 (48) max 6
                                                          summary (base 0) u 4 min 0 (0) max 3
                                                            summary (base 0) u 2 min 0 (0) max 1 (1)
                                                              cluster[0] (base 0) u 2 min 1 (1) max 1
                                                                cluster[1] (base 2) u 2 min 0 (2) max 1 (3)
                                                                  cluster[0] (base 48) u 2 min 1 (49) max 1
                                                                    cluster[1] (base 50) u 2 min 0 (50) max 0
                                                                      cluster[2] (base 52) u 2 min 0 (52) max 0
                                                                        cluster[3] (base 54) u 2 min 0 (54) max 0
                                                                          cluster[7] (base 56) u 8 min 0 (56) max 3
                                                                            summary (base 0) u 4 min 0 (0) max 1
                                                                              summary (base 0) u 2 min 0 (0) max 0
                                                                                cluster[0] (base 0) u 2 min 1 (1) max 1
                                                                                  cluster[0] (base 56) u 2 min 1 (57) max 1
                                                                                    cluster[1] (base 58) u 2 min 0 (58) max 1 (59)

```

Insert of 55 (110111) completed:

```

root (base 0) u 64 min 2 (2) max 59 [1: insert(55)]
  summary (base 0) u 8 min 0 (0) max 7
    summary (base 0) u 4 min 0 (0) max 3
      summary (base 0) u 2 min 0 (0) max 1 (1)
        cluster[0] (base 0) u 2 min 1 (1) max 1
          cluster[1] (base 2) u 2 min 0 (2) max 1 (3)
            cluster[0] (base 0) u 2 min 1 (1) max 1
              cluster[1] (base 2) u 2 min 0 (2) max 1 (3)
                cluster[2] (base 4) u 2 min 1 (5) max 1
                  cluster[3] (base 6) u 2 min 0 (6) max 1 (7)
                    cluster[0] (base 0) u 8 min 4 (4) max 7
                      summary (base 0) u 4 min 3 (3) max 3
                        cluster[3] (base 6) u 2 min 1 (7) max 1
                          cluster[1] (base 8) u 8 min 2 (10) max 7
                            summary (base 0) u 4 min 1 (1) max 3
                              summary (base 0) u 2 min 1 (1) max 1
                                cluster[1] (base 2) u 2 min 0 (2) max 1 (3)
                                  cluster[1] (base 10) u 2 min 1 (11) max 1
                                    cluster[2] (base 12) u 2 min 0 (12) max 1 (13)
                                      cluster[3] (base 14) u 2 min 0 (14) max 1 (15)
                                        cluster[2] (base 16) u 8 min 4 (20) max 4
                                          cluster[3] (base 24) u 8 min 2 (26) max 2
                                            cluster[5] (base 40) u 8 min 2 (42) max 7
                                              summary (base 0) u 4 min 2 (2) max 3
                                                summary (base 0) u 2 min 1 (1) max 1
                                                  cluster[1] (base 2) u 2 min 1 (3) max 1
                                                    cluster[2] (base 44) u 2 min 1 (45) max 1
                                                      cluster[3] (base 46) u 2 min 0 (46) max 1 (47)
                                                        cluster[6] (base 48) u 8 min 0 (48) max 7 [2: minimum=0
                                                            3: insert into non-empty sub-cluster
                                                            4: insert(7)
                                                            9: increasing V->max to 7]
                                                          summary (base 0) u 4 min 0 (0) max 3
                                                            summary (base 0) u 2 min 0 (0) max 1 (1)
                                                              cluster[0] (base 0) u 2 min 1 (1) max 1
                                                                cluster[1] (base 2) u 2 min 0 (2) max 1 (3)
                                                                  cluster[0] (base 48) u 2 min 1 (49) max 1
                                                                    cluster[1] (base 50) u 2 min 0 (50) max 0
                                                                      cluster[2] (base 52) u 2 min 0 (52) max 0
                                                                        cluster[3] (base 54) u 2 min 0 (54) max 1 (55) [5: minimum=0
                                                                            6: insert into non-empty
                                                                              sub-cluster
                                                                              7: insert(1)
                                                                              8: increasing V->max to 1]
                                                                          cluster[7] (base 56) u 8 min 0 (56) max 3
                                                                            summary (base 0) u 4 min 0 (0) max 1
                                                                              summary (base 0) u 2 min 0 (0) max 0
                                                                                cluster[0] (base 0) u 2 min 1 (1) max 1
                                                                                  cluster[0] (base 56) u 2 min 1 (57) max 1
                                                                                    cluster[1] (base 58) u 2 min 0 (58) max 1 (59)

```

Delete of 42 (101010) completed:

```

root (base 0) u 64 min 2 (2) max 59 [1: delete(42)
                                2: deleting 42 from sub-cluster]
summary (base 0) u 8 min 0 (0) max 7
  summary (base 0) u 4 min 0 (0) max 3
    summary (base 0) u 2 min 0 (0) max 1 (1)
    cluster[0] (base 0) u 2 min 1 (1) max 1
    cluster[1] (base 2) u 2 min 0 (2) max 1 (3)
  cluster[0] (base 0) u 2 min 1 (1) max 1
  cluster[1] (base 2) u 2 min 0 (2) max 1 (3)
  cluster[2] (base 4) u 2 min 1 (5) max 1
  cluster[3] (base 6) u 2 min 0 (6) max 1 (7)
cluster[0] (base 0) u 8 min 4 (4) max 7
  summary (base 0) u 4 min 3 (3) max 3
  cluster[3] (base 6) u 2 min 1 (7) max 1
cluster[1] (base 8) u 8 min 2 (10) max 7
  summary (base 0) u 4 min 1 (1) max 3
    summary (base 0) u 2 min 1 (1) max 1
    cluster[1] (base 2) u 2 min 0 (2) max 1 (3)
  cluster[1] (base 10) u 2 min 1 (11) max 1
  cluster[2] (base 12) u 2 min 0 (12) max 1 (13)
  cluster[3] (base 14) u 2 min 0 (14) max 1 (15)
cluster[2] (base 16) u 8 min 4 (20) max 4
cluster[3] (base 24) u 8 min 2 (26) max 2
cluster[5] (base 40) u 8 min 5 (45) max 7 [3: delete(2)
                                           4: cluster losing minimum
                                           7: V->min replaced by 5
                                           8: deleting 5 from sub-cluster
                                           12: sub-cluster for 5 is empty
                                           28: minimum=5]
  summary (base 0) u 4 min 3 (3) max 3 [5: minimum=2
                                           13: delete(2)
                                           14: cluster losing minimum
                                           17: V->min replaced by 3
                                           18: deleting 3 from sub-cluster
                                           22: sub-cluster for 3 is empty
                                           25: need new max to replace 3
                                           27: new max not found, now a one-element
                                           cluster]
    summary (base 0) u 2 min / max / [15: minimum=1
                                           23: delete(1)
                                           24: cluster is losing its one member
                                           26: maximum=-1]
    cluster[1] (base 2) u 2 min / max / [16: minimum=1
                                           19: delete(1)
                                           20: cluster is losing its one member
                                           21: minimum=-1]
  cluster[2] (base 44) u 2 min / max / [6: minimum=1
                                           9: delete(1)
                                           10: cluster is losing its one member
                                           11: minimum=-1]
    cluster[3] (base 46) u 2 min 0 (46) max 1 (47)
cluster[6] (base 48) u 8 min 0 (48) max 7
  summary (base 0) u 4 min 0 (0) max 3
    summary (base 0) u 2 min 0 (0) max 1 (1)
    cluster[0] (base 0) u 2 min 1 (1) max 1
    cluster[1] (base 2) u 2 min 0 (2) max 1 (3)
  cluster[0] (base 48) u 2 min 1 (49) max 1
  cluster[1] (base 50) u 2 min 0 (50) max 0
  cluster[2] (base 52) u 2 min 0 (52) max 0
  cluster[3] (base 54) u 2 min 0 (54) max 1 (55)
cluster[7] (base 56) u 8 min 0 (56) max 3
  summary (base 0) u 4 min 0 (0) max 1
    summary (base 0) u 2 min 0 (0) max 0
    cluster[0] (base 0) u 2 min 1 (1) max 1
  cluster[0] (base 56) u 2 min 1 (57) max 1
  cluster[1] (base 58) u 2 min 0 (58) max 1 (59)

```

Delete of 55 (110111) completed:

```

root (base 0) u 64 min 2 (2) max 59 [1: delete(55)
                                2: deleting 55 from sub-cluster]
  summary (base 0) u 8 min 0 (0) max 7
    summary (base 0) u 4 min 0 (0) max 3
      summary (base 0) u 2 min 0 (0) max 1 (1)
        cluster[0] (base 0) u 2 min 1 (1) max 1
        cluster[1] (base 2) u 2 min 0 (2) max 1 (3)
      cluster[0] (base 0) u 2 min 1 (1) max 1
      cluster[1] (base 2) u 2 min 0 (2) max 1 (3)
      cluster[2] (base 4) u 2 min 1 (5) max 1
      cluster[3] (base 6) u 2 min 0 (6) max 1 (7)
  cluster[0] (base 0) u 8 min 4 (4) max 7
    summary (base 0) u 4 min 3 (3) max 3
      cluster[3] (base 6) u 2 min 1 (7) max 1
  cluster[1] (base 8) u 8 min 2 (10) max 7
    summary (base 0) u 4 min 1 (1) max 3
      summary (base 0) u 2 min 1 (1) max 1
        cluster[1] (base 2) u 2 min 0 (2) max 1 (3)
      cluster[1] (base 10) u 2 min 1 (11) max 1
      cluster[2] (base 12) u 2 min 0 (12) max 1 (13)
      cluster[3] (base 14) u 2 min 0 (14) max 1 (15)
  cluster[2] (base 16) u 8 min 4 (20) max 4
  cluster[3] (base 24) u 8 min 2 (26) max 2
  cluster[5] (base 40) u 8 min 5 (45) max 7
    summary (base 0) u 4 min 3 (3) max 3
      cluster[3] (base 46) u 2 min 0 (46) max 1 (47)
  cluster[6] (base 48) u 8 min 0 (48) max 6 [3: delete(7)
                                          4: deleting 7 from sub-cluster
                                          9: corrected V->max to 6
                                          10: minimum=0]
    summary (base 0) u 4 min 0 (0) max 3
      summary (base 0) u 2 min 0 (0) max 1 (1)
        cluster[0] (base 0) u 2 min 1 (1) max 1
        cluster[1] (base 2) u 2 min 0 (2) max 1 (3)
      cluster[0] (base 48) u 2 min 1 (49) max 1
      cluster[1] (base 50) u 2 min 0 (50) max 0
      cluster[2] (base 52) u 2 min 0 (52) max 0
      cluster[3] (base 54) u 2 min 0 (54) max 0 [5: delete(1)
                                                6: leaf cluster going from two
                                                members to one
                                                7: minimum=0
                                                8: maximum=0]
  cluster[7] (base 56) u 8 min 0 (56) max 3
    summary (base 0) u 4 min 0 (0) max 1
      summary (base 0) u 2 min 0 (0) max 0
        cluster[0] (base 0) u 2 min 1 (1) max 1
      cluster[0] (base 56) u 2 min 1 (57) max 1
      cluster[1] (base 58) u 2 min 0 (58) max 1 (59)

```



## Test 1, Summer 2010

Fill in the min and max blanks for the following instance of a van Emde Boas tree for the set  $\{0, 1, 8, 10, 11, 12, 13\}$ . You should give these as values in the local universe  $(0 \dots u-1)$ . Instead of using the symbol “/” for NIL, use the symbol “ $\emptyset$ ”. (10 points)

```

root (base 0) u 16 min _____ max _____
  summary (base 0) u 4 min _____ max _____
    summary (base 0) u 2 min _____ max _____
      cluster[0] (base 0) u 2 min _____ max _____
      cluster[1] (base 2) u 2 min _____ max _____
    cluster[0] (base 0) u 4 min _____ max _____
      summary (base 0) u 2 min _____ max _____
        cluster[0] (base 0) u 2 min _____ max _____
        cluster[1] (base 2) u 2 min _____ max _____
      cluster[1] (base 4) u 4 min _____ max _____
        summary (base 0) u 2 min _____ max _____
          cluster[0] (base 4) u 2 min _____ max _____
          cluster[1] (base 6) u 2 min _____ max _____
        cluster[2] (base 8) u 4 min _____ max _____
          summary (base 0) u 2 min _____ max _____
            cluster[0] (base 8) u 2 min _____ max _____
            cluster[1] (base 10) u 2 min _____ max _____
          cluster[3] (base 12) u 4 min _____ max _____
            summary (base 0) u 2 min _____ max _____
              cluster[0] (base 12) u 2 min _____ max _____
              cluster[1] (base 14) u 2 min _____ max _____

```