Three Structurally Similar List Update Upper Bound Proofs

Review of amortized complexity:

$$\hat{c}_i = c_i + \Phi(i) - \Phi(i-1)$$

$$c_i = \hat{c}_i + \Phi(i-1) + \Phi(i)$$

$$\sum_{i=1}^{n} c_i = \sum_{i=1}^{n} \hat{c}_i + \Phi(0) - \Phi(n)$$

If $\Phi(0) - \Phi(n)$ is never positive, then an upper bound on $\sum_{i=1}^{n} \hat{c}_i$ is an upper bound on $\sum_{i=1}^{n} c_i$.

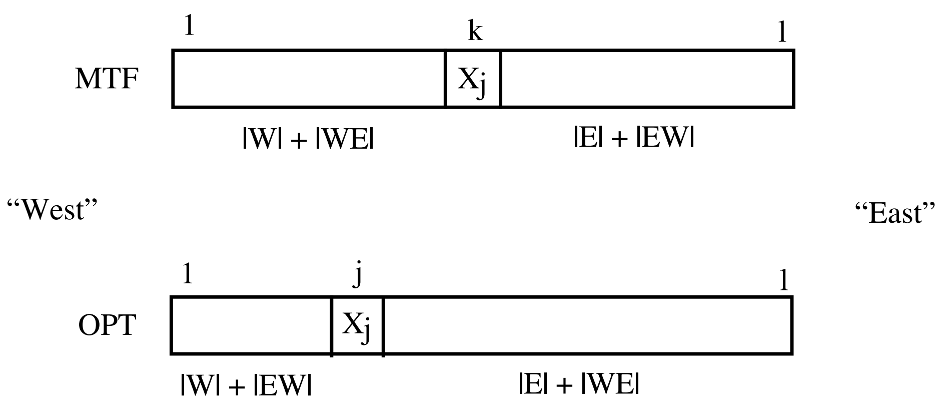
1. To compare MTF to OPT (without detailing OPT), the amortized cost of each MTF operation is bounded.

Φ = Number of inversions (two elements in different order) between MTF list and OPT list.

Initially the lists are the same, so $\Phi(0) = 0$, $\Phi(i) \geq 0$, and $\Phi(0) - \Phi(n) \leq 0$.

What happens when MTF and OPT process the same request, even though the lists are different?

Before processing ACCESS to X_j



W = Items “west” of X_j in both lists

E = Items “east” of X_j in both lists

WE = Items west of X_j in MTF, but east of X_j in OPT (* and *v* in book)

EW = Items east of X_j in MTF, but west of X_j in OPT

Observations:

$j = |W| + |EW| + 1$

$k = |W| + |WE| + 1$

$|W| = k - 1 - |WE|$

$k - 1 - |WE| + |EW| + 1 = j$

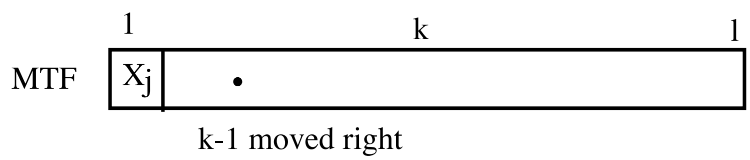
$k - |WE| \leq j$

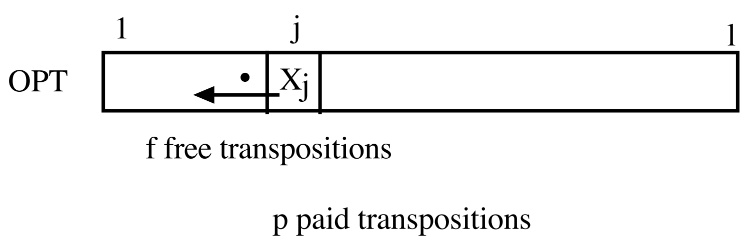
Amortized cost of ACCESS of X_j (by MTF)

Search + $|W|$ new inversions - $|WE|$ lost inversions

$$k \quad + \quad (k - 1 - |WE|) \quad - \quad |WE| \quad = \quad 2(k - |WE|) - 1 \quad \leq \quad 2j - 1$$

BUT, OPT list also changes





Actual cost = $j + p$

"Correction" to Δ in $\Phi \leq -f$ for lost inversions $+ p$ for new inversions

So, after accessing and updating both lists, the amortized cost of MTF ACCESS is

$$k + k - 1 - |WE| - |WE| - f + p \leq 2j - 1 - f + p \leq 2OPT - 1$$

For a sequence σ of n accesses

$$MTF(\sigma) \le 2OPT_C(\sigma) - n - OPT_F(\sigma) + OPT_P(\sigma)$$

where $OPT_C(\sigma)$ is cost without paid transposes, $OPT_F(\sigma)$ is the free transposes (already in $OPT_C(\sigma)$), and $OPT_P(\sigma)$ is the paid transposes. (Ex. 1.2. shows that free transposes are not critical.)

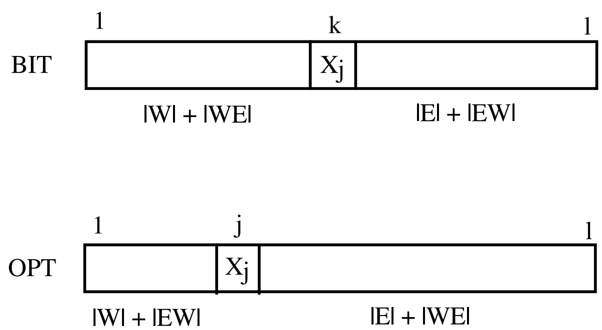
$$MTF(\sigma) \le 2OPT_C(\sigma) - n + OPT_P(\sigma)$$
$$= OPT(\sigma) + OPT_C(\sigma) - n \le 2OPT(\sigma) - n$$

2. To compare BIT to OPT, the *expected* amortized cost of each BIT operation is bounded.

   Φ = Sum of *weighted* inversions (two elements in different order) between BIT list and OPT list.

   Suppose pair appears with x before y in BIT list and y before x in OPT list.

   Weight of inversion is the value of y's bit + 1



(Expected) Amortized cost of ACCESS of X_j (by BIT)

Search + 1/2 Cost if MTF occurs + 1/2 Cost if MTF does not occur =
(bit for X_j is now 1) (bit for X_j is now 0)

Search + 1/2(Weight of new inversions for W less the weight of lost inversions for WE)
+ 1/2 Decrease in weight for WE

MTF: (expected) weight of each new inversion (BIT: X_j, w; OPT: w, X_j) for W is 3/2

MTF: weight of each lost inversion for WE is 1 (didn't adjust Φ for bit change from 0 to 1)

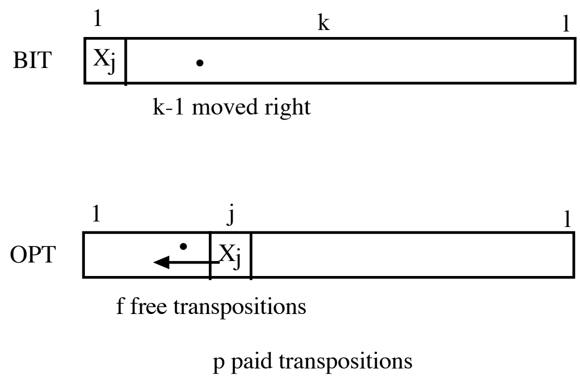
No MTR: The decrease in weight for each WE inversion will be 1

$$k + 1/2(3/2|W| - |WE|) + 1/2(-|WE|) = k + 3/4|W| - |WE| = k + 3/4(k - 1 - |WE|) - |WE|$$
$$= 7/4k - 3/4 - 7/4|WE| \le 7/4j - 3/4$$

BUT, OPT also searches and changes its list

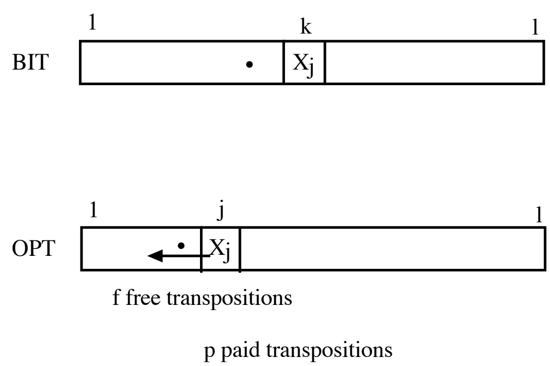
Actual cost = $j + p$

With probability 1/2, BIT changed X_j bit from 0 to 1 (so MTF occured)



"Correction" to Δ in $\Phi \le 1/2(-3/2f + 3/2p) = -3/4f + 3/4p$

With probability 1/2, BIT changed X_j bit from 1 to 0 (so no MTF occured)



"Correction" to Δ in $\Phi \le 1/2(f + 3/2p) = 1/2f + 3/4p$

This gives an overall expected correction $\le -1/4f + 3/2p$

Applying correction

$$
\begin{aligned}
7/4k - 3/4 - 7/4|WE| -1/4f + 3/2p &\le 7/4j - 3/4 - 1/4f + 3/2p \\
&\le 7/4j + 7/4p - 3/4 \\
&= 7/4OPT - 3/4
\end{aligned}
$$

3. To compare RMTF to OPT, the *expected* amortized cost of each RMTF operation is bounded.

Since MTF is decided by a coin flip, the expected number of accesses to an element before a MTF is 2:

$$\sum_{i=0}^{\infty} \left(\frac{1}{2}\right)^{i} = \frac{1}{1-\frac{1}{2}} = 2$$

But if the element is not MTF'd, the expectation remains 2.

Amortized cost of ACCESS of X_j (by RMTF)

Search + 1/2 Cost if MTF occurs + 1/2 Cost if MTF does not occur =

Search + 1/2(Weight of new inversions for W less the weight of lost inversions for WE)
+ 1/2 • 0

$k + 1/2(2|W| - 2|WE|) = k + |W| - |WE| = k + (k - 1 - (|WE|) - |WE|$

$= 2k - 1 - 2|WE| \leq 2j - 1$

BUT, OPT also searches and changes its list

Actual cost = j + p

With probability 1/2, MTF occured

"Correction" to Δ in $\Phi \leq 1/2(-2f + 2p) = -f + p$

With probability 1/2, no MTF occured

"Correction" to Δ in $\Phi \leq 1/2(2f + 2p) = f + p$

This gives an overall expected correction $\leq 2p$

Applying correction

$2k - 1 - 2|WE| + 2p \leq 2j - 1 + 2p \leq 2OPT - 1$