CSE 2315 - Discrete Structures

Homework 4- Solution - Fall 2010

Due Date: Nov. 11 2010, 3:30 pm

Relations

1. Indicate which of the listed ordered pairs are part of the given relations \( \rho \) on \( \mathbb{N} \) (16 points)

   a) \( x \rho y \leftrightarrow y = x^2 + x \) : (1, 3), (1, 2), (3, 10), (4, 20), (4, 15), (5, 30)
      (1, 2), (4, 20), (5, 30)

   b) \( x \rho y \leftrightarrow x^2 + y = 27 \) : (2, 5), (3, 3), (4, 7), (4, 11), (5, 2), (5, 6)
      (4, 11), (5, 2)

   c) \( x \rho y \leftrightarrow x \text{ is even} \land x^3 \leq y \) : (1, 1), (1, 2), (2, 3), (2, 8), (2, 11), (3, 30)
      (2, 8), (2, 11)

   d) \( x \rho y \leftrightarrow y^2 = 25 \) : (2, 4), (2, 5), (3, 7), (3, 9), (4, 5), (7, 5)
      (2, 5), (4, 5), (7, 5)

Rewrite the following relations on the set \( S = \{2, 5, 6, 9\} \) as a set of ordered pairs.

   e) \( x \rho y \leftrightarrow x - y = 3 \)
      \( \{(5, 2), (9, 6)\} \)

   f) \( x \rho y \leftrightarrow x \ast y > 10 \)
      \( \{(2, 6), (2, 9), (5, 5), (5, 6), (5, 9), (6, 2), (6, 5), (6, 6), (6, 9), (9, 2), (9, 5), (9, 6), (9, 9)\} \)

   g) \( x \rho y \leftrightarrow (x + y) \text{ is odd} \land x > y \)
      \( \{(5, 2), (6, 5), (9, 2), (9, 6)\} \)

   h) \( x \rho y \leftrightarrow (x \ast y) \text{ is even} \land x \text{ is a prime number} \)
      \( \{(2, 2), (2, 5), (2, 6), (2, 9), (5, 2), (5, 6)\} \)

2. Indicate for each of the following binary relations on the positive integers if they are one-to-one, one-to-many, many-to-one, or many-to-many. Also determine if the relations are reflexive, symmetric, antisymmetric, or transitive. (9 points)

   a) \( \rho = \{(1, 2), (2, 3), (2, 4), (3, 6), (4, 5)\} \) many-to-one

   b) \( x \rho y \leftrightarrow x \ast y = 36 \) one-to-one

   c) \( \rho = \{(1, 1), (1, 3), (1, 6), (2, 2), (3, 3), (3, 6), (6, 6)\} \) many-to-many
4. Draw the Hasse diagrams for the following partial orderings on \( S = \{1, 2, 3, 4, 5, 6\} \). Also list the least, greatest, minimal, and maximal elements. (12 + 8 points)

a) \( \rho = \{ (1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6), (1, 2), (2, 4), (1, 4), (4, 6), (2, 6), (3, 5), (1, 6) \} \)

There is no least element but 1 and 3 are the minimal elements.
There is no greatest element but 5 and 6 are the maximal elements.

b) \( \rho = \{ (1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6), (2, 3), (3, 5), (2, 5) \} \)
There is no least element but 1, 2, 4 and 6 are the minimal elements.
There is no greatest element but 1, 4, 5, and 6 are the maximal elements.

c) \( \rho = \{(1,1), (2,2), (3,3), (4,4), (5,5), (6,6), (4,2), (6,4), (5,3), (6,2), (3,1), (5,1), (2,4)\} \)

Is not a partial ordering since it is not antisymmetric. As a result there is not corresponding Hasse diagram.

If the last element is removed, the resulting relation \( \rho = \{(1,1), (2,2), (3,3), (4,4), (5,5), (6,6), (4,2), (6,4), (5,3), (6,2), (3,1), (5,1)\} \)
has the following diagram

There is no least element but 5 and 6 are the minimal elements.
There is no greatest element but 1 and 2 are the maximal elements.

d) \( \rho = \{(1,1), (2,2), (3,3), (4,4), (5,5), (6,6), (1,2), (2,4), (1,4), (2,6), (4,6), (3,6), (1,6), (5,6), (1,3)\} \)

There is no least element but 1 and 5 are the minimal elements.
6 is the greatest and maximal element.

List all the equivalent classes of the following equivalence relations on the set \( S = \{1, 2, 3, 4, 5\} \).

e) \( \rho = \{(1,1), (2,2), (3,3), (4,4), (5,5), (1,2), (2,3), (2,5), (5,1), (1,3), (5,3), (3,2), (3,1), (1,5), (2,1), (3,5), (5,2)\} \)

\[ [1] = \{1, 2, 3, 5\} \]
\[ [4] = \{4\} \]

f) \( \rho = \{(1,1), (2,2), (3,3), (4,4), (5,5), (3,5), (2,4), (5,3), (4,2)\} \)

\[ [1] = \{1\} \]
\[ [2] = \{2, 4\} \]
\[ [3] = \{3, 5\} \]
g) \( \rho = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (1, 3), (3, 2), (4, 5), (2, 1), (3, 1), (2, 3), (5, 4), (1, 2)\} \)

\[ [1] = \{1, 2, 3\} \]
\[ [4] = \{4, 5\} \]

h) \( \rho = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (2, 4), (4, 2)\} \)

\[ [1] = \{1\} \]
\[ [2] = \{2, 4\} \]
\[ [3] = \{3\} \]
\[ [5] = \{5\} \]

**Functions**

5. Indicate if the following functions are one-to-one, onto, or bijective. If a function is a bijection also find its inverse. (9 points)

a) \( f : \mathbb{Z} \rightarrow \mathbb{R} , \ f(x) = x^2 + 5 \)

has none of the properties

b) \( f : \mathbb{N} \rightarrow \mathbb{Z} , \ f(x) = (-1)^x \cdot \lceil \frac{x}{2} \rceil \)

If \( \mathbb{N} \) includes 0 it is a bijection. Otherwise it is injective (one-to-one) (but not surjective - onto). The inverse (for the first case) is \( f^{-1}(x) = 2 \cdot |x| - \frac{x + |x|}{2|x|} \)

c) \( f : \{1, 2, 3, 5\} \rightarrow \{4, 6, 8, 9\} , \ f = \{(1, 4), (2, 6), (3, 8), (5, 9)\} \)

is bijective. Its inverse is \( f^{-1} = \{(4, 1), (6, 2), (8, 3), (9, 5)\} \)

d) \( f : \{a, b, c, d, e\} \rightarrow \{\text{vowel, consonant}\} , \ f = \{(a, \text{vowel}), (b, \text{consonant}), (c, \text{consonant}), (d, \text{consonant}), (e, \text{vowel})\} \)

is surjective (onto)

e) \( f : \mathbb{R} \rightarrow \mathbb{R} , \ f(x) = x - 7 \)

is bijective. Its inverse is \( f^{-1}(x) = x + 7 \)

f) \( f : (\mathbb{N} \cup \{0\}) \times \{0, 1\} \rightarrow \mathbb{Z} , \ f((x, y)) = x \cdot (y - 1) \)

has none of the properties

6. Rewrite the following permutations on \( S = \{1, 2, 3, 4, 5, 6\} \) in cycle notation. (10 points)

a) \( f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 5 & 1 & 4 & 6 \end{pmatrix} \)

\( f = (1, 2, 3, 5, 4) \)

b) \( f = \{(1, 2), (2, 4), (3, 3), (4, 1), (5, 6), (6, 5)\} \)

\( f = (1, 2, 4) \circ (5, 6) \)

c) \( f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 2 & 4 & 6 & 1 & 3 \end{pmatrix} \)

\( f = (1, 5) \circ (3, 4, 6) \)
d) \( f = \{(1, 1), (2, 6), (3, 3), (4, 2), (5, 5), (6, 4)\} \)
   \( f = (2, 6, 4) \)

\[ f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 5 & 1 & 6 & 2 & 4 \end{pmatrix} \]
\( f = (1, 3) \circ (2, 5) \circ (4, 6) \)

7. Prove by giving appropriate constants that the following functions have the indicated orders of magnitude. (18 points)

a) \( f(x) = \sin(x^2 - \pi) \), \( f = \Theta(1) \)
   This can strictly not be shown since there is no positive constant to show the lower bound.
   But we can show \( f = O(1) \) with \( c = 1, n_0 = 0 \)

b) \( f(x) = 2 \times x^3 - x + 100 \), \( f = O(x^3) \)
   \( c = 2, n_0 = 100 \)

c) \( f(x) = \frac{x^2 - 7x + 5}{4x^2 - 3} \), \( f = \Theta(x) \)
   \( c_1 = \frac{1}{5}, c_2 = \frac{1}{5}, n_0 = 32 \)

d) \( f(x) = 4 \times x^3 - 7 \times x + 2^x \), \( f = O(2^x) \)
   \( c = 2, n_0 = 14 \)

e) \( f(x) = \log(x + x^2) \), \( f = O(\log(x)) \)
   \( c = 3, n_0 = 2 \)

f) \( f(x) = \log(3^x + x^2) \), \( f = \Theta(x) \)
   \( c_1 = \log(3), c_2 = 1, n_0 = 1 \)