Algorithms and Data Structures

Discrete Structures and Math Review

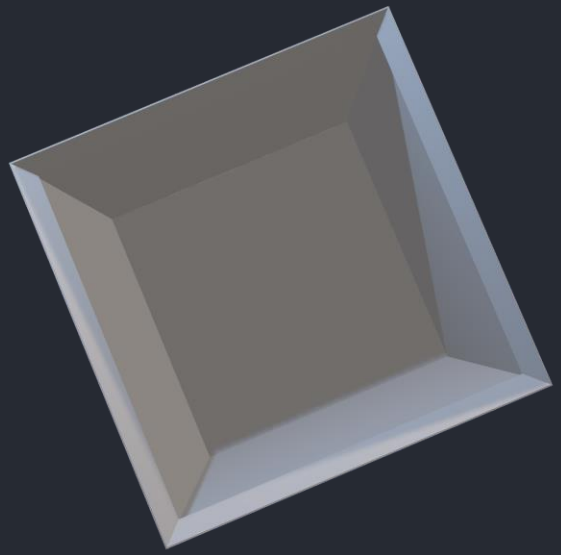
A set is a collection of objects.

$A =$ 

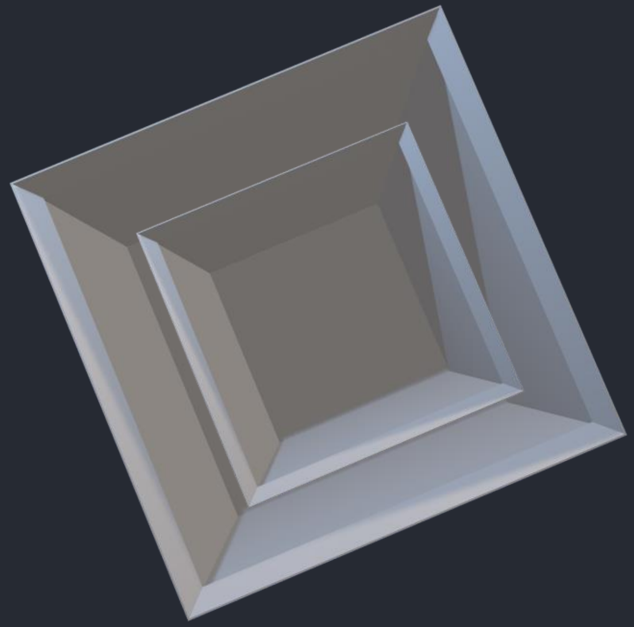
A set is a collection of objects.

👀 $\in A$. ✅ $\in A$. 👻 $\in A$.

😴 $\notin A$. 📅 $\notin A$. 8️⃣ $\notin A$.

$A =$ 

Empty Set / Null Set

$A =$ 

Set containing the empty set

A = {😃, ⭐, 🥳, 🏠, 🫎}

B = {⭐, 🥳, 🐯, 🥇}

$A = \{$ 😃, ⭐, 🥳, 🏠, 🫎 $\}$

$B = \{$ ⭐, 🥳, 🐯, 🥇 $\}$

$A \cap B = \{$ ⭐, 🥳 $\}$

A = {😃, ⭐, 🥳, 🏠, 🫎}

B = {⭐, 🥳, 🐯, 🥇}

$A \cup B$ = {😃, ⭐, 🥳, 🏠, 🫎, 🐯, 🥇}

Graphs

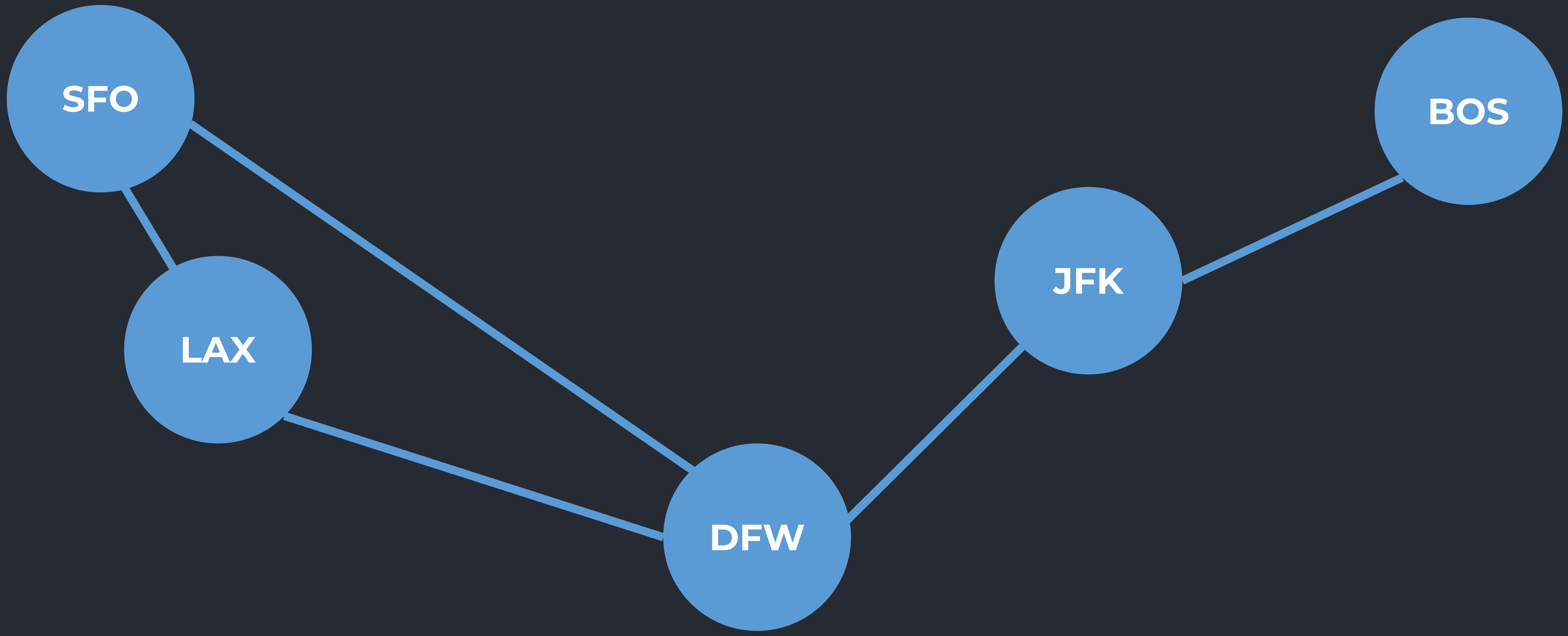
Formal Definition

$$G = (V, E)$$

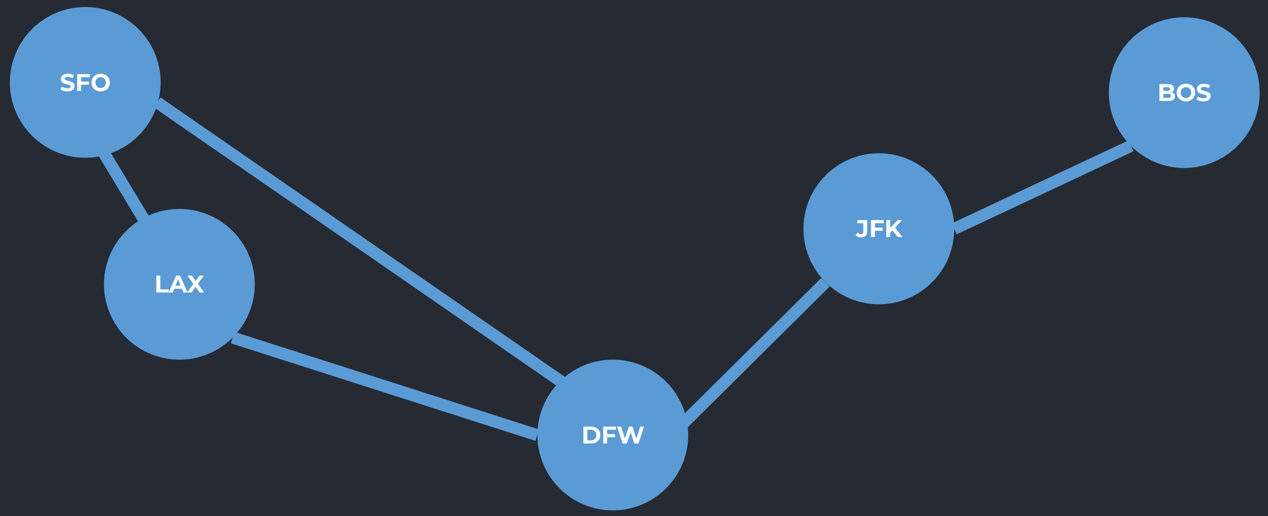
V: Set of vertices

E: Set of edges

Graphs



Graphs - Undirected

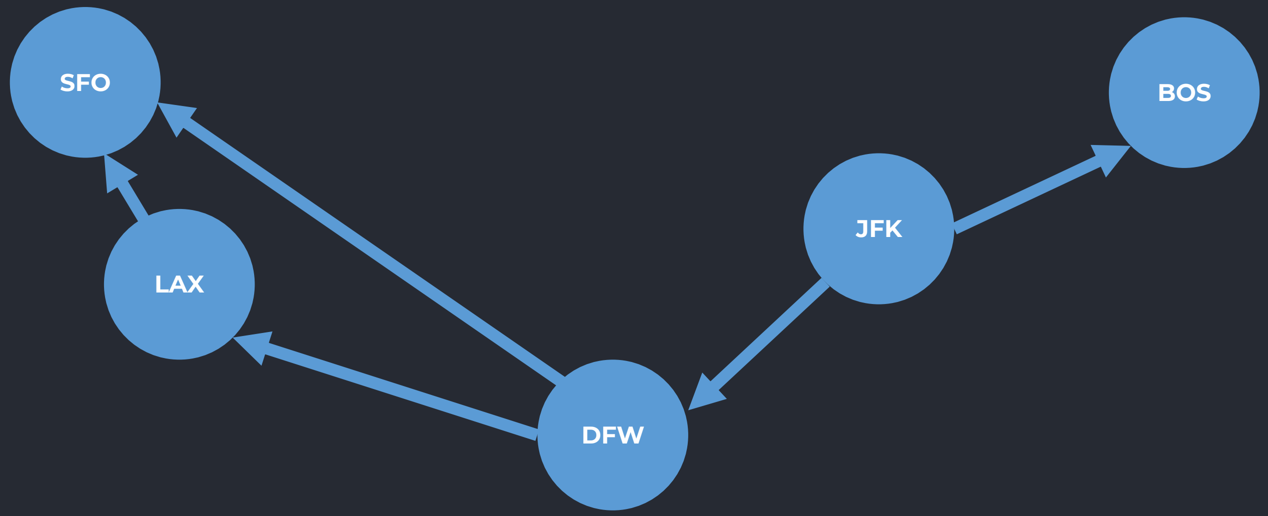


$$G = (V, E)$$

$$V = \{SFO, LAX, DFW, JFK, BOS\}$$

$$E = \{\{SFO, LAX\}, \{SFO, DFW\}, \{LAX, DFW\}, \{DFW, JFK\}, \{JFK, BOS\}\}$$

Graphs - Directed

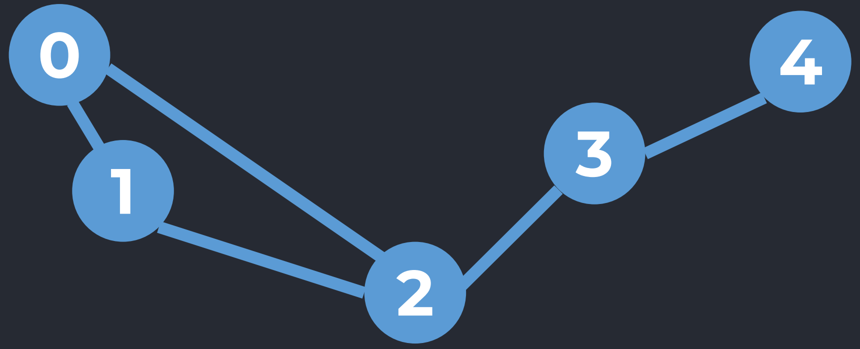


$$G = (V, E)$$

$$V = \{SFO, LAX, DFW, JFK, BOS\}$$

$$E = \{(DFW, LAX), (DFW, SFO), (LAX, SFO), (JFK, DFW), (JFK, BOS)\}$$

Graphs - Representation

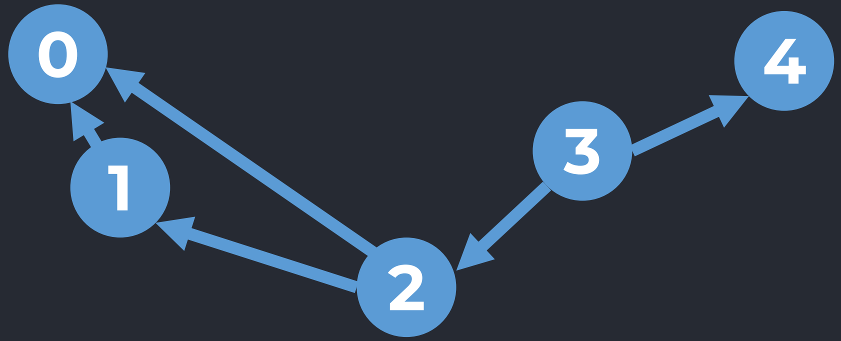


To:

From:	0	1	2	3	4
0	0	1	1	0	0
1	1	0	1	0	0
2	1	1	0	1	0
3	0	0	1	0	1
4	0	0	0	1	0

Adjacency Matrix

Graphs - Representation



To:

From:	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	0
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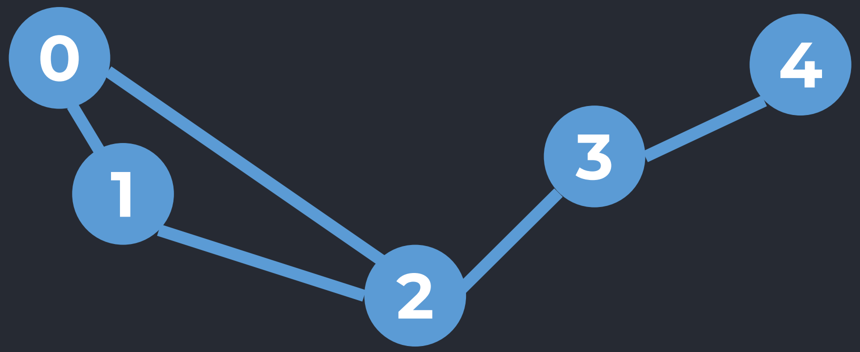
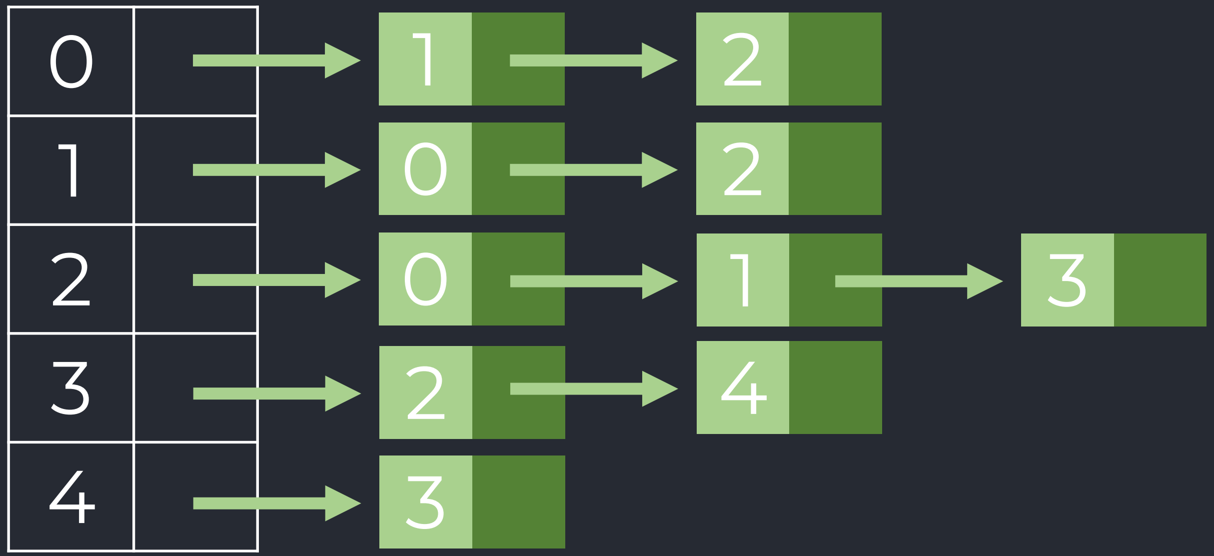
Adjacency Matrix

Graphs - Representation





Adjacency List

Summations

$$\sum_{i=p}^{q} a_i$$

Index Variable i

Initial Value p

Final Value q

📒 Summations

$$\sum_{i=p}^{q} a_i$$

Summations

$$\sum_{i=p}^{q} a_i = a_p + a_{p+1} + \cdots + a_{q-1} + a_q$$

📒 Summations

$$\sum_{i=1}^{6} i = 1 + 2 + 3 + 4 + 5 + 6$$

📒 Summations

$$\sum_{i=1}^{6} i^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2$$

Summations

$$\sum_{k=1}^{6} k^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2$$

The choice of index variable does not change the value of the sum.

📒 Exercise

$$\sum_{n=3}^{4} n^2 = 3^2 + 4^2 = 25$$

📈 Sequences

Intuitively,

A *sequence* is an **ordered list** of objects.

1, 2, 3, 4, 5, 7, ...
1, 1, 2, 3, 5, ...
a, aa, aaa, aaaa, aaaaa, ...

Sequences

More formally,

A *sequence* is a function whose domain is the set of natural numbers ($\mathbb{N}$).

1, 3, 5, 7, 9, ...

$f(1) = 1$

$f(2) = 3$

$f(3) = 5$

$f(4) = 7$

Sequences

More formally,

A *sequence* is a function whose domain is the set of natural numbers ($\mathbb{N}$).

1, 3, 5, 7, 9, ...

$a_1 = 1$

$a_2 = 3$

$a_3 = 5$

$a_4 = 7$

Sequences

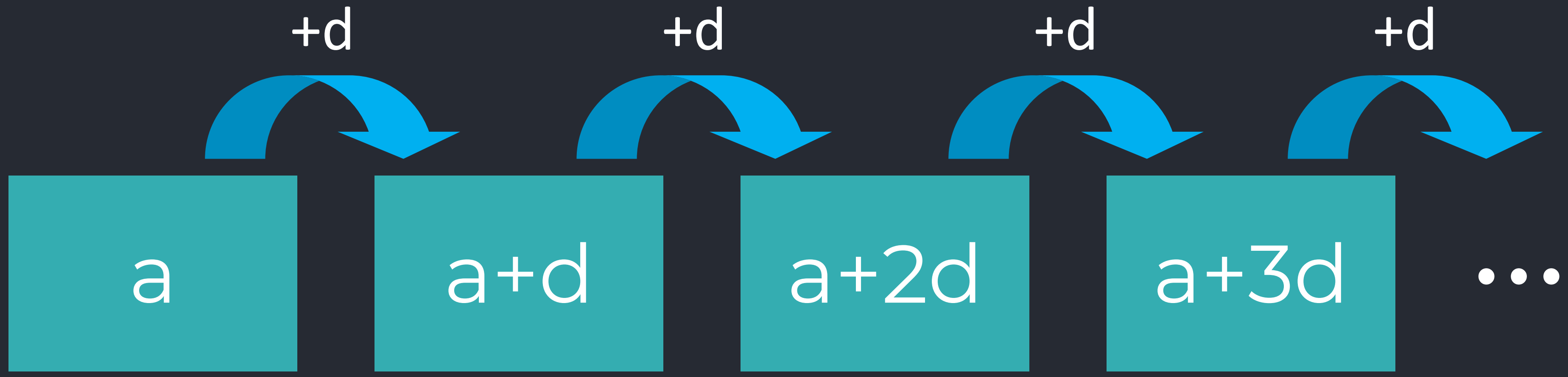
Definition.

An *arithmetic sequence* is a sequence where the difference between consecutive terms always remains the same.

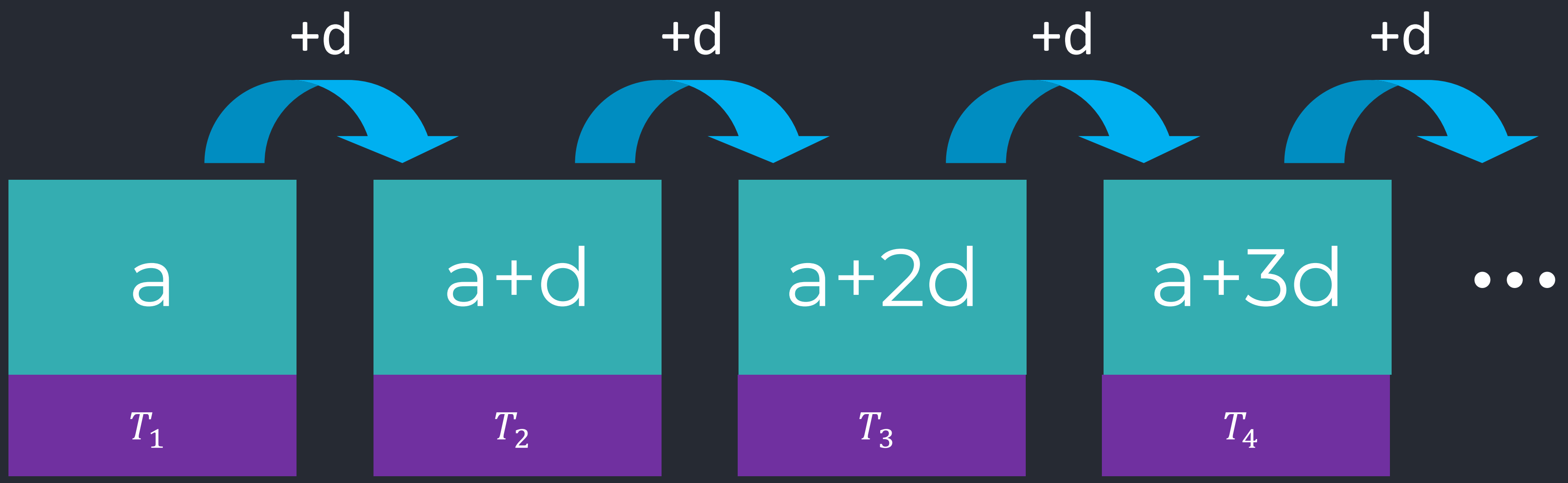
Sequences

Uniqueness

Any arithmetic sequence can be uniquely characterized by its first term (a) and the common difference (d).



Sequences



$$T_n = a + (n - 1)d$$

Arithmetic Sequences

$$T_n = a + (n-1)d$$

$$T_5 = a + 4d$$

$$T_{10} = a + 9d$$

$$T_{20} = a + 19d$$

Arithmetic Sequences

$$T_n = a + (n-1)d$$

Suppose we have an arithmetic sequence. Can we find a formula for its sum to n terms?

$$S_n = \sum_{i=1}^{n} (a + (i-1)d)$$

Arithmetic Sequences

Suppose we have an arithmetic sequence. Can we find a formula for its sum to n terms?

$$S_n = \sum_{i=1}^{n} (a + (i-1)d)$$

Story time 😄

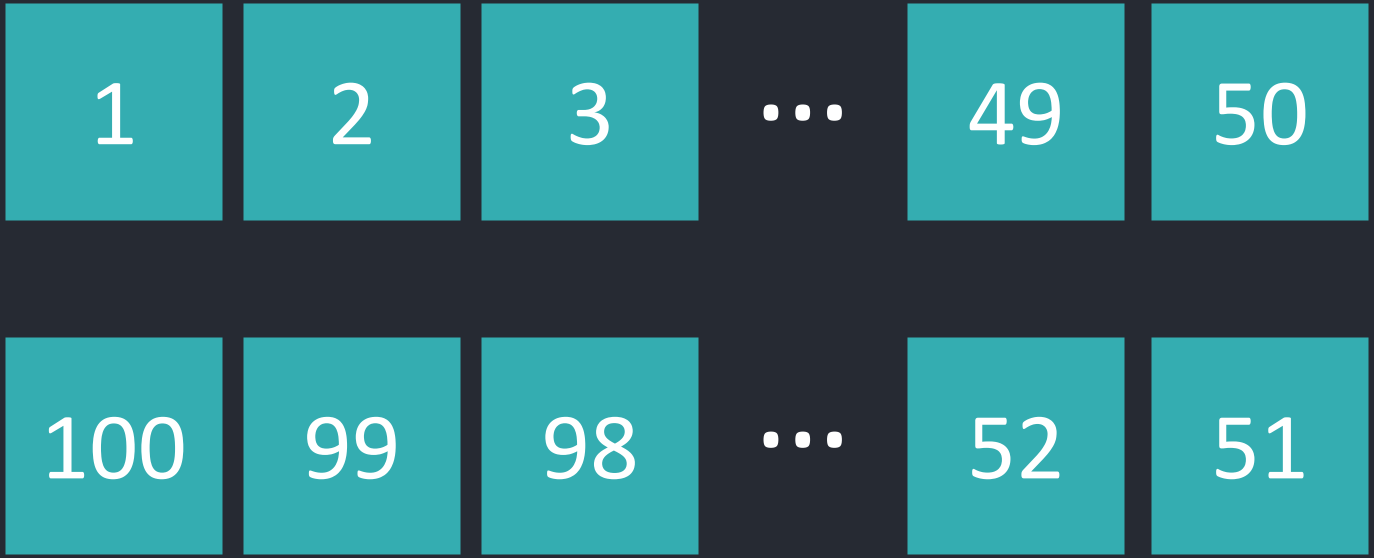
$$1 + 2 + 3 + \cdots + 98 + 99 + 100$$

It’s 5050.

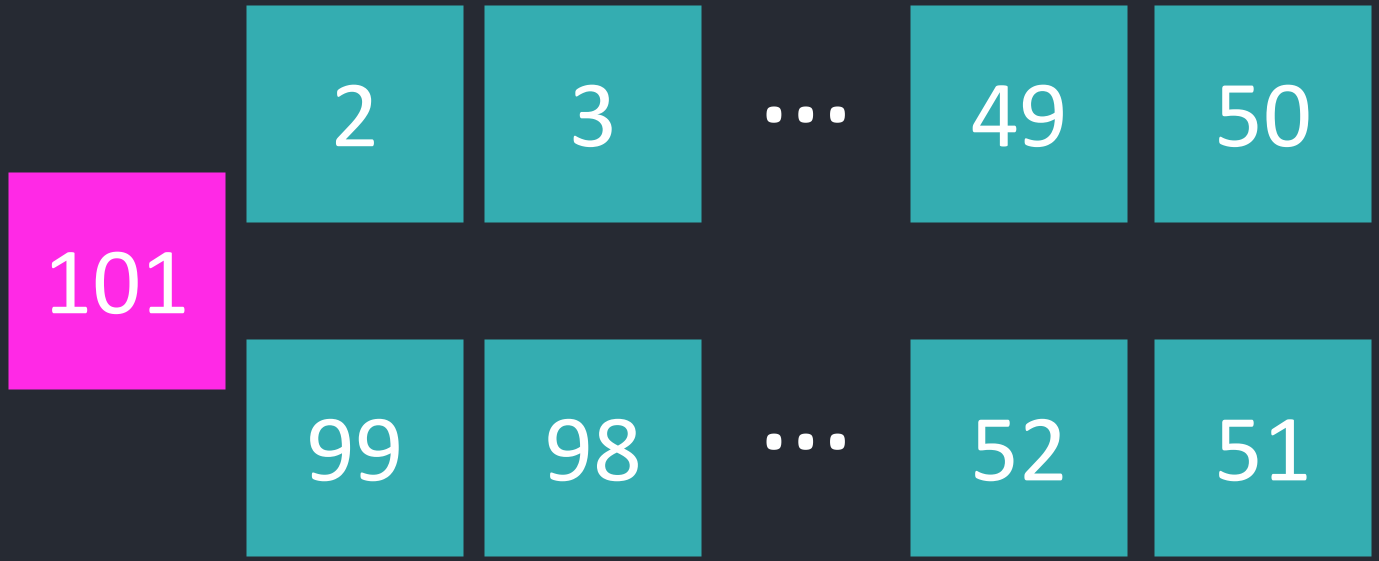
I can prove it.

There’s no way you’re right! Go check your work again.

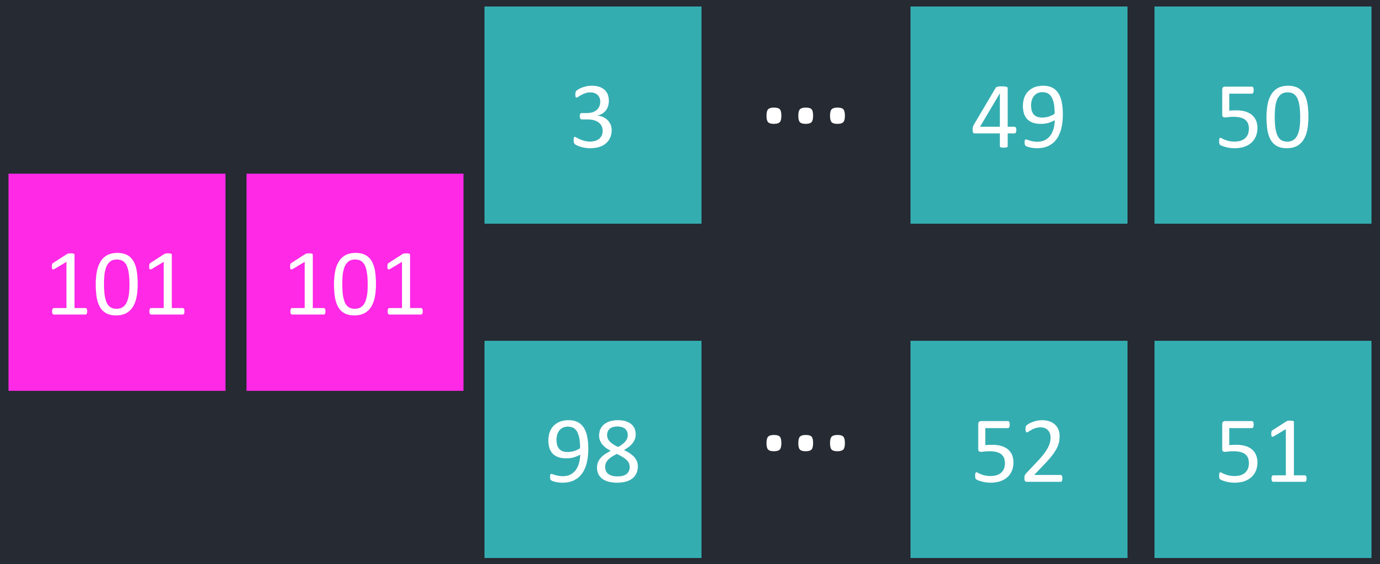
$$1 + 2 + 3 + \cdots + 49 + 50 + 51 + 52 \cdots + 98 + 99 + 100$$



$$1 + 2 + 3 + \cdots + 49 + 50 + 51 + 52 \cdots + 98 + 99 + 100$$



$$1 + 2 + 3 + \cdots + 49 + 50 + 51 + 52 \cdots + 98 + 99 + 100$$



$$1 + 2 + 3 + \cdots + 49 + 50 + 51 + 52 \cdots + 98 + 99 + 100$$

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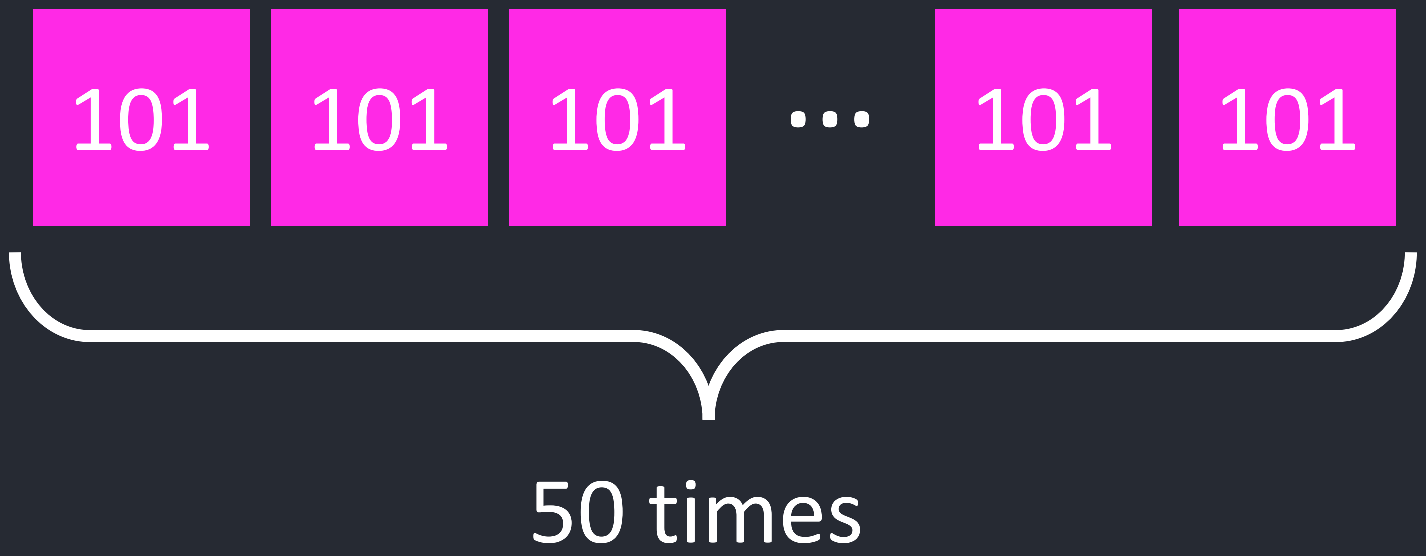
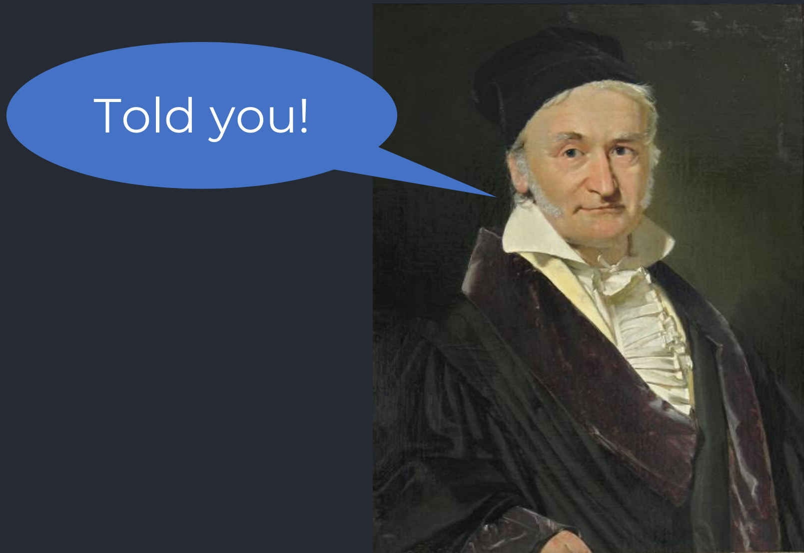


$$1 + 2 + 3 + \cdots + 49 + 50 + 51 + 52 \cdots + 98 + 99 + 100$$

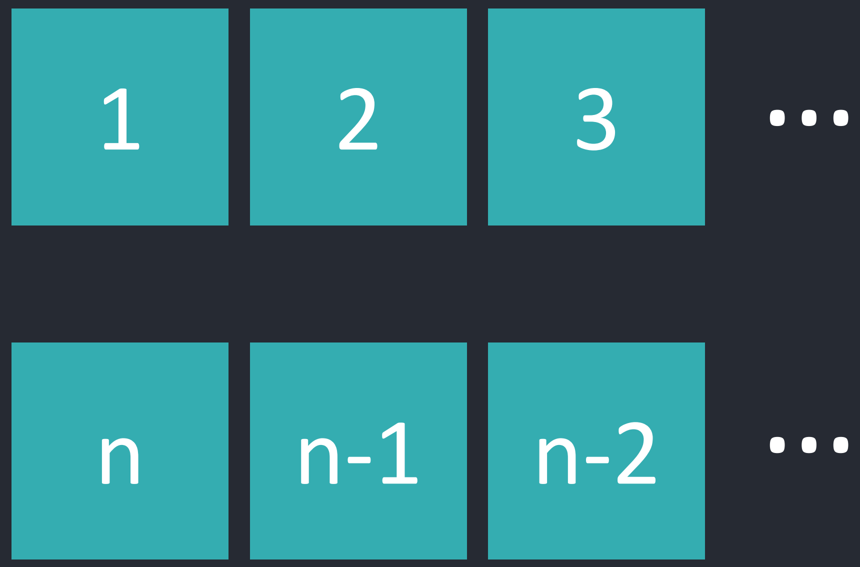
50 * 101

$$1 + 2 + 3 + \cdots + 49 + 50 + 51 + 52 \cdots + 98 + 99 + 100$$

5050



Generalizing to the First n Natural Numbers,



Generalizing to the First n Natural Numbers,

$$\underbrace{n+1 + n+1 + n+1 + \dots}_{\frac{n}{2}\text{ times}}$$

Generalizing to the First n Natural Numbers

$$\sum_{i=1}^{n} i = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

Key Insight 💡

In a finite arithmetic sequence, the sum of the i^{th} term from the start and the i^{th} term from the end remains the same, for all i.

Generalizing to Arithmetic Series

$$S_n = \sum_{i=1}^{n} (a + (i-1)d)$$

a	$a + d$	$a + 2d$	…
$a + (n-1)d$	$a + (n-2)d$	$a + (n-3)d$	…

Generalizing to Arithmetic Series

$$S_n = \sum_{i=1}^{n} (a + (i-1)d)$$

$2a + (n-1)d$

$a + d$ | $a + 2d$ ...

$a + (n-2)d$ | $a + (n-3)d$ ...

Generalizing to Arithmetic Series

$$S_n = \sum_{i=1}^{n} (a + (i-1)d)$$

$2a + (n-1)d$ $2a + (n-1)d$

$a + 2d$ ...

$a + (n-3)d$ ...

Generalizing to Arithmetic Series

$$S_n = \sum_{i=1}^{n} (a + (i-1)d)$$

$2a + (n-1)d$	$2a + (n-1)d$	$2a + (n-1)d$	…

Generalizing to Arithmetic Series

$$S_n = \sum_{i=1}^{n} (a + (i-1)d)$$

$2a + (n-1)d$	$2a + (n-1)d$	$2a + (n-1)d$	...

$\frac{n}{2}$ times

Generalizing to Arithmetic Series

$$S_n = \sum_{i=1}^{n} (a + (i-1)d)$$

$$= \frac{n}{2} \cdot (2a + (n-1)d)$$

Generalizing to Arithmetic Series

$$S_n = \sum_{i=1}^{n} (a + (i-1)d)$$

$$= \frac{n}{2} \cdot (\text{ First Term} + \text{Last Term })$$

Sequences

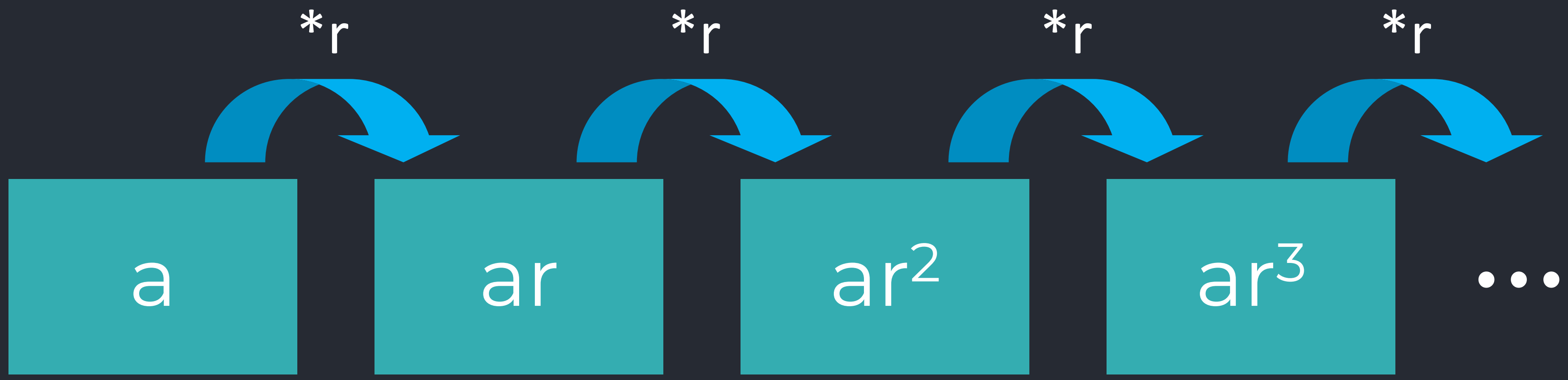
Definition.

A *geometric sequence* is a sequence where the ratio of consecutive terms always remains the same.

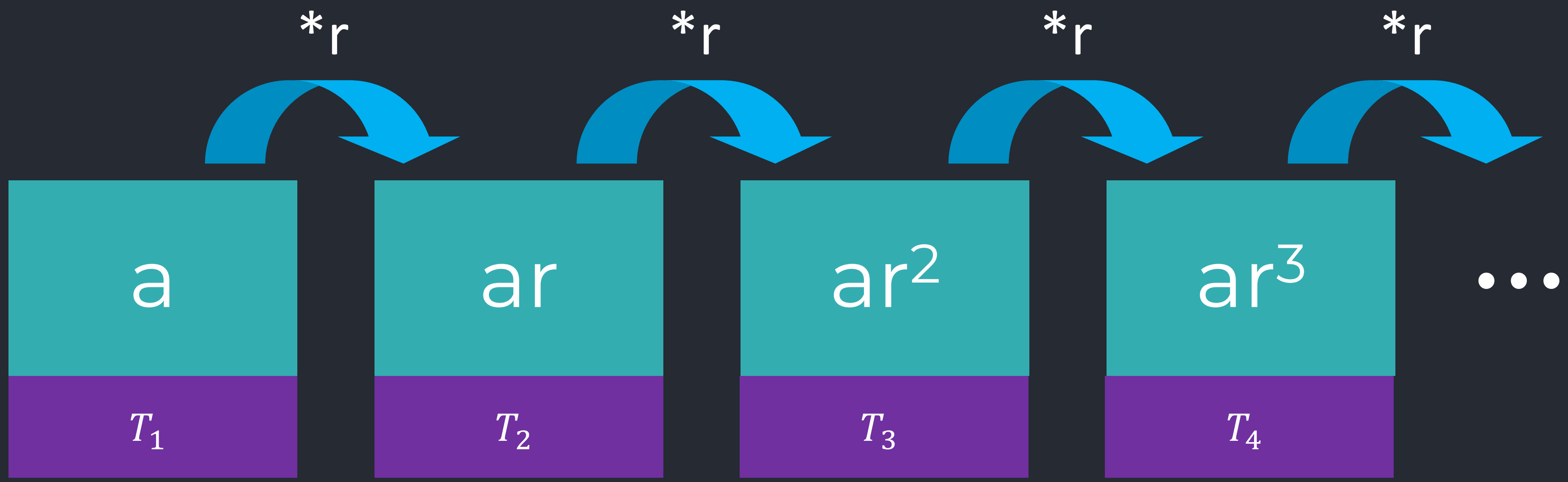
Sequences

Uniqueness

Any geometric sequence can be uniquely characterized by its first term (a) and the common ratio (r).



Sequences



$$T_n = ar^{n-1}$$

Sum to n terms of a Geometric Sequence

$$S_n = a + ar + ar^2 + ar^3 + \cdots + ar^{n-2} + \mathrm{ar}^{n-1}$$

$$r \cdot S_n = r\ (a + \ ar + ar^2 + ar^3 + \cdots + ar^{n-2} + \mathrm{ar}^{n-1})$$

Sum to n terms of a Geometric Sequence

$$S_n = a + ar + ar^2 + ar^3 + \cdots + ar^{n-2} + ar^{n-1}$$

$$- \quad r \cdot S_n = \quad ar + ar^2 + ar^3 + \cdots + ar^{n-2} + ar^{n-1} + ar^n$$

$$S_n(1 - r) = a - ar^n$$

Sum to n terms of a Geometric Sequence

$$S_n(1-r) = a - ar^n$$

Sum to n terms of a Geometric Sequence

$$S_n = \frac{a - ar^n}{(1 - r)}$$

Sum to n terms of a Geometric Sequence

$$a + ar + ar^2 + \cdots + ar^{n-1} = \frac{a - ar^n}{(1 - r)}$$

Sum to n terms of a Geometric Sequence

$$a + ar + ar^2 + \cdots + ar^{n-1} = \frac{\text{First Term} - \text{First excluded term}}{(1 - r)}$$

⏱️ Time Complexity

Measuring Running Time of Code

- It is easy to simply measure how long it takes for a program to execute using a computer's clock.
- But is that a good idea?

Other processes running may affect measured runtime.

Runtime is system-dependent

Instead, let’s count the number of “basic operations” our algorithm performs.