Sorting Algorithms Properties

Insertion Sort

Binary Search

CSE 3318 – Algorithms and Data Structures
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Summary

- Properties of sorting algorithms
- Sorting algorithms
  - Insertion sort – Chapter 2 (CLRS)
- Indirect sorting - (Sedgewick Ch. 6.8 'Index and Pointer Sorting')
- Binary Search
  - See the notation conventions (e.g. $\log_2 N = \lg N$)
- Terminology and notation:
  - $\log_2 N = \lg N$
  - Use interchangeably:
    - *Runtime* and *time complexity*
    - *Record* and *item*

Sorting

Sorting

- Sort an array, **A**, of items (numbers, strings, etc.).
- Why sort it?
  - To use in binary search.
  - To compute rankings, statistics (min/max, top-10, top-100, median).
  - Check that there are no duplicates
  - intersection and union are easier to perform between 2 sorted sets
  - ….
- We will study several sorting algorithms,
  - Pros/cons, behavior .
- Insertion sort
- Optional, self study: selection sort.

Properties of sorting

- Stable:
  - It does not change the relative order of items whose keys are equal.

- Adaptive:
  - The time complexity will depend on the input
    - E.g. if the input data is almost sorted, it will run significantly faster than if not sorted.
    - see later insertion sort vs selection sort.

Other aspects of sorting

- Time complexity: worst/best/average

- Number of data moves: copy/swap the DATA RECORDS
  - One data move = 1 copy operation of a complete data record
  - Data moves are NOT updates of variables independent of record size (e.g. loop counter)

- Space complexity: Extra Memory used
  - Do NOT count the space needed to hold the INPUT data, only extra space (e.g. copy of data)
  - *Θ(1):* ***In place*** methods: constant extra memory
  - *Θ(N):* Uses extra space proportional to the number of items:
    - For pointers (e.g. linked lists or indirect access)
    - For a copy of the data

- Direct vs indirect sorting
  - Direct: move items as needed to sort
  - Indirect: move *pointers/handles* to items.
    - Can keep the key with pointer or not.

- Later: non-comparison sorting

Stable sorting

- A ***sorting algorithm is stable*** iff, after it sorts an array, any two records that compare equal, will still be in the same relative order as they were before sorting and this happens ***for every*** possible ***input array***.
- Example:
  - An item consists of an int (e.g. GPA) and a string (e.g. name).
  - Sort based on: GPA (integer)

4 *Bob*	3 Tom	*4* *Anna*	3 Jane	1 Henry

- Stable sort (OK: Tom before Jane and Bob before Anna):

1 Henry	3 Tom	3 Jane	*4* *Bob*	*4* *Anna*

- Unstable sort (violation: Anna is now before Bob):

1 Henry	3 Tom	3 Jane	*4* *Anna*	*4* *Bob*

- ***Note: Stable is a property of the algorithm, NOT of the algorithm-data pair***. You CANNOT say "This algorithm is stable for this input". It must be so for all inputs.

Stable sorting - Application

- Applications
  - Sorting by 2 criteria,
    - E.g.: 1st by GPA, 2nd by name:
      - When the GPA is the same, have data in order of names
    - Solution:
      - First sort by name (with any method)
      - Next, with a stable sort, sort by GPA
    - Alternative solution:
      - write a more complex comparison function.
  - Part of other sorting methods
    - See later: LSD radix sort uses a stable sort (count sort).

Proving an Algorithm is Stable

- An algorithm ***is stable*** if we can guarantee/prove that this property happens for any input (not just a few example inputs).

  => To prove it, must use an actual proof (possibly using a loop invariant) or give a very good explanation. Checking that "it works" on a few examples is NOT a proof. It must work for every possible input that is valid.

- An algorithm ***is not stable*** if there is at least one possible input for which it breaks the property.

  => To prove it, find one example input for which the property fails.

  => easier to prove.

- *Intuition: if an algorithm swaps items that are away from each other (jump over other items) it is most likely NOT stable.*
  - This statement is a guideline, not a proof. Make sure you always *find an example* if you suspect this case.

Insertion sort

Insertion sort

Process the array from left to right.
Step j (outer loop):
- elements A[0],A[1],…A[j-1] are already sorted
- insert element A[j] in it's place among A[0],..A[j-1] (inner loop)

Each row shows the array after one iteration of the outer loop (after step j).

original	5	3	7	8	5	0	4
1st	**3**	5	7	8	5	0	4
2nd	3	5	**7**	8	5	0	4
3rd	3	5	7	**8**	5	0	4
4th	3	5	**5**	7	8	0	4
5th	**0**	3	5	5	7	8	4
6th	0	3	**4**	5	5	7	8

Elements in shaded cells are sorted, but they have only items that were originally in the shaded cells. They are not in final position (e.g. see the 8 move all the way to the right).

- See TedEd video
- Wikipedia (see "A graphical example of insertion sort"): https://en.wikipedia.org/wiki/Insertion_sort
- Brief and nice resource: http://interactivepython.org/runestone/static/pythonds/SortSearch/TheInsertionSort.html
- Animation for version that swaps elements: https://youtu.be/Q1JdRUh1_98 (sent by Aryan)

Insertion Sort

```
void insertion_sort(int A[],int N){
  int j,k,curr;
  for (j=1; j<N; j++){
    curr = A[j];
  // insert curr (A[j]) in the
  // sorted sequence A[0…j-1]
    k = j-1;
    while ((k>=0) && (A[k]>curr)){
       A[k+1] = A[k];
       k = k-1;
    }
    A[k+1] = curr;
}
```

'Data move' is an assignment.
(matters if deep copy or pointer is used)
Each red number: 2 moves.
Each blue number: 1 move.
Best: Θ(N)
Worst: $\Theta(N^2)$ Average: $\Theta(N^2)$

5	3	7	8	5	0	4
3	5	7	8	5	0	4
3	5	**7**	8	5	0	4
3	5	7	**8**	5	0	4
3	5	**5**	7	8	0	4
0	3	5	5	7	8	4
0	3	**4**	5	5	7	8

j	Inner loop time complexity: Best : 1	Worst: j	Average: j/2
1	1	1	1/2
2	1	2	2/2
…	…		
N-2	1	N-2	(N-2)/2
N-1	1	N-1	(N-1)/2
Θ	Θ(N)	$\Theta(N^2)$	$\Theta(N^2)$

Repetition of while-k
At most: j (Includes end loop check)
At least: 1 (Evaluate:(k>0 and A[k]>key))

Insertion Sort Time Complexity

j	Inner loop time complexity: Best : 1	Worst: j	Average: j/2
1	1	1	1
2	1	2	2/2
…	…		
N-2	1	N-2	(N-2)/2
N-1	1	N-1	(N-1)/2
Total	(N-1)	[N * (N-1)]/2	[N * (N-1)]/4
Order of magnitude	**Θ(N)**	**Θ(N^2)**	**Θ(N^2)**
Data that produces it.	Sorted	Sorted in reverse order	Random data

```
void insertion_sort(int A[],int N){
  int j,k, curr;
  for (j=1; j<N; j++){
  curr = A[j];
  // insert curr (A[j]) in the
  // sorted sequence A[0…j-1]
  k = j-1;
  while ((k>=0) && (A[k]>curr)){
     A[k+1] = A[k];
     k = k-1;
  }
  A[k+1] = curr;
}
```

Insertion sort is adaptive

=> $O(N^2)$

'Total' instructions in worst case:

(N-1) + (N-2) + … 2 + 1 =

= [N * (N-1)]/2 -> **Θ(N^2)**

Note that the N^2 came from the summation, NOT because 'there is an N in the inner loop' (NOT because N * N).

"O" will be explained in detail later. It says that the algorithm take at most order of N^2.

See the Khan Academy for a discussion on the use of O(N2): https://www.khanacademy.org/computing/computer-science/algorithms/insertion-sort/a/insertion-sort

Insertion sort: Time Complexity & Data moves

Each row shows the array after one iteration of the outer loop for each algorithm.

'Data move' is an assignment. (Implementation will matter: deep copy or pointer)

Each row shows the array after one iteration of the outer loop for each algorithm.

Time complexity

Insert the next element in it's place in the sorted sequence to the left of it.

original	5	3	7	8	5	0	4
1st	**3**	**5**	7	8	5	0	4
2nd	3	5	**7**	8	5	0	4
3rd	3	5	7	**8**	5	0	4
4th	3	5	**5**	7	8	0	4
5th	**0**	3	5	5	7	8	4
6th	0	3	**4**	5	5	7	8

Gray cells are visited by the iterations of the inner loop => they are proportional with the time complexity => ~ $N^2/2$ (worst case)

Data moves

Insert the next element in it's place in the sorted sequence to the left of it.

5	3	7	8	5	0	4
3	**5**	7	8	5	0	4
3	5	**7**	8	5	0	4
3	5	7	**8**	5	0	4
3	5	**5**	**7**	**8**	0	4
0	**3**	**5**	**5**	**7**	**8**	4
0	3	**4**	**5**	**5**	**7**	**8**

Each red number: 2 moves.
Each blue number: 1 move.
Best: 2(N-1) Worst: 2(N-1)+N(N-1)/2
Average:2N+N(N-1)/4

Insertion sort - Properties

- Time complexity: $O(N^2)$ ($\Theta(N)$ – best, $\Theta(N^2)$ – worst and average)
- Space complexity: $\Theta(1)$ (it does not copy any of array)
- Data moves: $\Theta(N)$ – best, $\Theta(N^2)$ – worst and average
- Adaptive: Yes ($\Theta(N)$ – best case, $\Theta(N^2)$ – worst and average case)
- Stable – Yes
- Direct sorting

Insertion sort - Variations

- Note how an algorithm has the capability to be stable but the way it is implemented can still make it unstable.
  - What happens if we use *A[k]>=key* in line 5?

- Give an implementation that uses a sentinel (to avoid the *k>0* check in line 5)
  - What is a sentinel?
  - Is it still stable? (How do you update/move the sentinel)?
  - Time complexity trade-off:
    - Cost to set-up the sentinel (linear: find the smallest element in the array) vs
    - Savings from removing the k>0 check (quadratic, in worst case).

Proving That an Algorithm is Correct

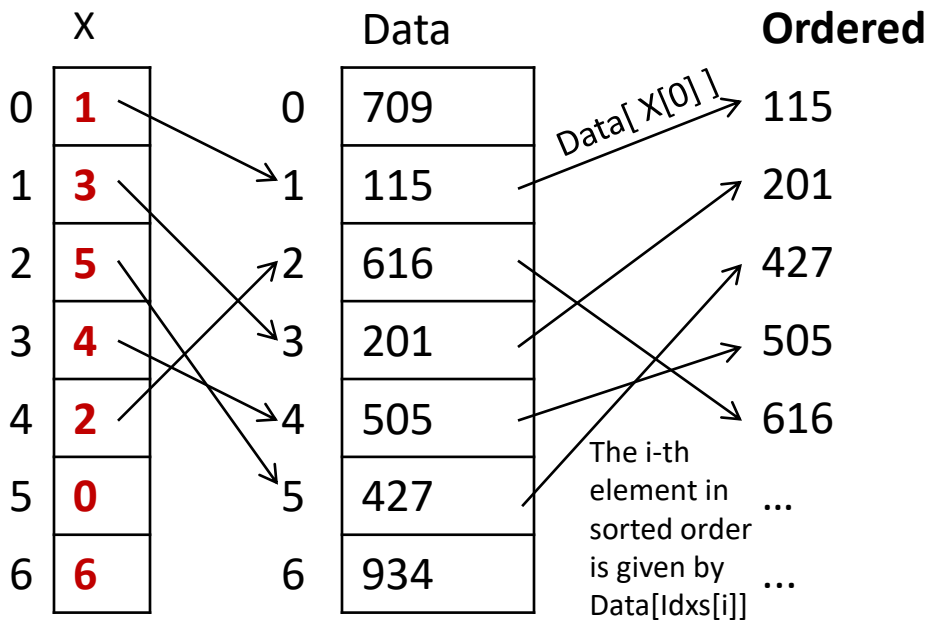
- **Required. Read the relevant book section if needed.**
- See CLRS, (starting at page 18), for proof of loop invariant and correctness of the insertion sort algorithm:
  - Identify a property that is preserved (maintained, built) by the algorithm: "the loop invariant" (b.c. preserved by the loop)
    - Which loop would you use here?
  - Show:
    - Initialization: it is true prior to the 1st iteration
    - Maintenance (j->j+1): If it is true before an iteration it remains true before the next iteration
    - Termination – use that property/invariant to show that the algorithm is correct
- What would the loop invariant be for the inner loop for insertion sort?
  - This question may be part of your next homework or quiz.

Indirect Sorting

- What if we need access to our data (array/list) in sorted order but
  - **do not want to move** records around
    - records are *too big*
    - records *are already sorted* by another criterion and we need that too.
  - **cannot move** records around.
    - *only have read access*, but no write access

- Solution: **Indirect sorting**
  - Generate a new array with references (e.g. indexes) to records in sorted order.

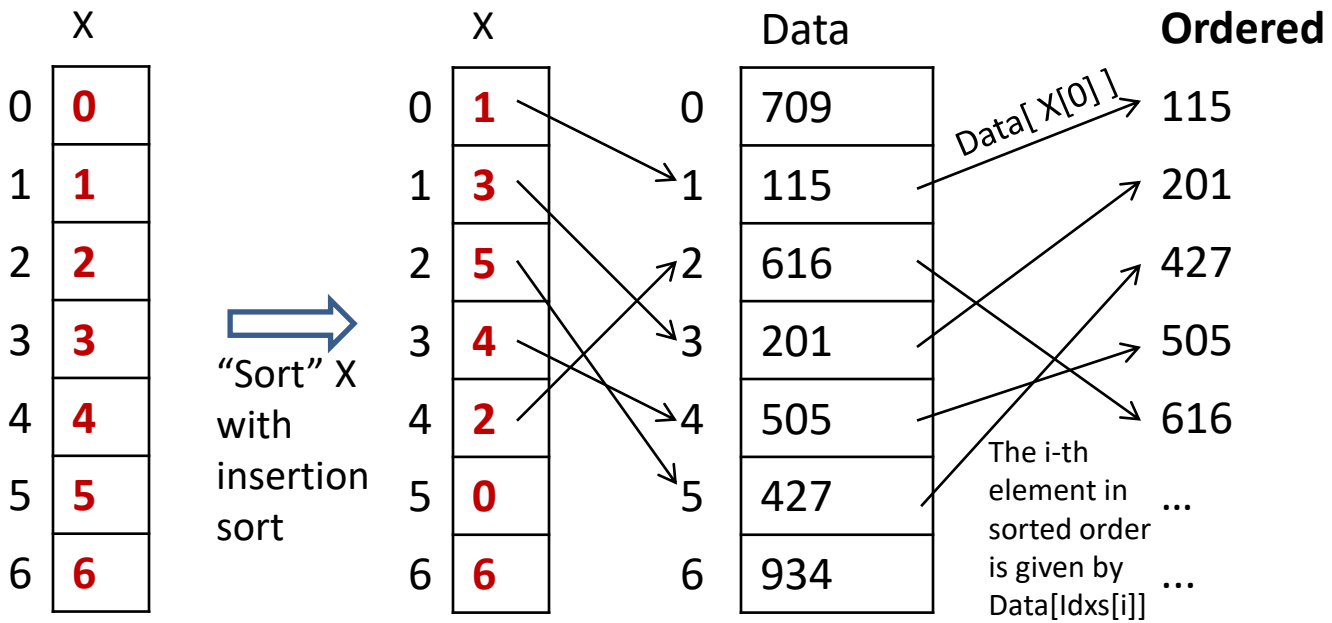
Indirect Sorting



j	X[j]	Data[X[j]]

```
for (j=0; j<7; j++){
    printf("%d\n", Data[ X[j] ] );
}
```

Indirect Sorting




```
void insertion_sort(int A[],int N){
  int j,k,curr;
  for (j=1; j<N; j++){
    curr = A[j];
    k = j-1;
    while ((k>=0) && ( A[k] > curr) ){
       A[k+1] = A[k];
       k = k-1;
    }
    A[k+1] = curr;
}
```

1. Create the identity array, X, with indexes 0 to N-1
2. Adapt insertion sort to rearrange the elements of X
    1. What do you copy?
    2. What do you compare?
3. Return X
    1. Language specific issues (C/Java)

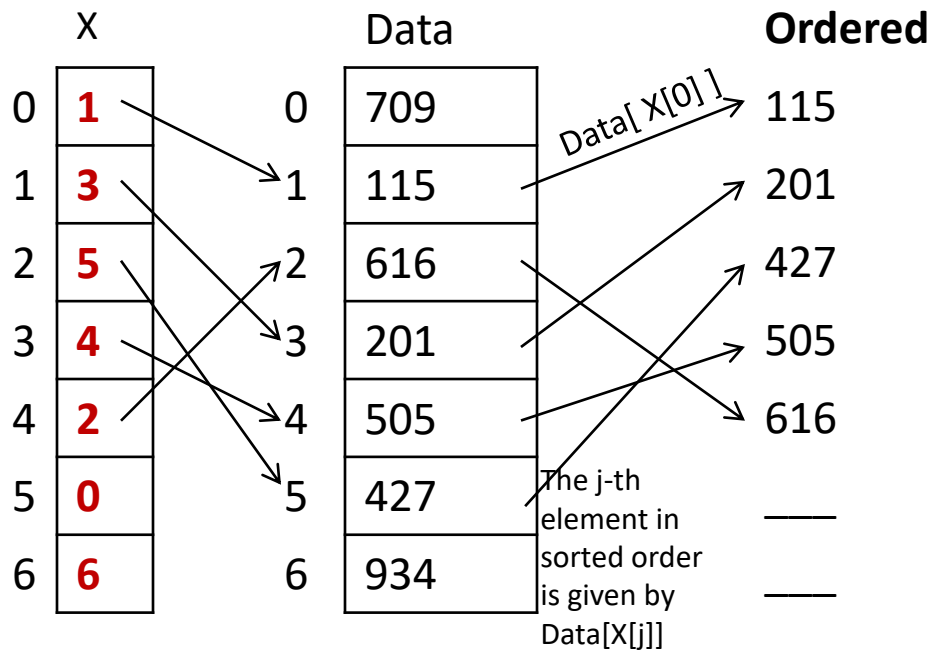
Access Data through X: e.g. Data[X[j]]

Indirect Sorting

- Food for thought:
  - Example of references:
    - indexes
    - memory addresses
    - offsets in a file
  - Can we indirect sort a linked list?

Binary Search and Indirect Sorting

Binary Search and Indirect Sorting

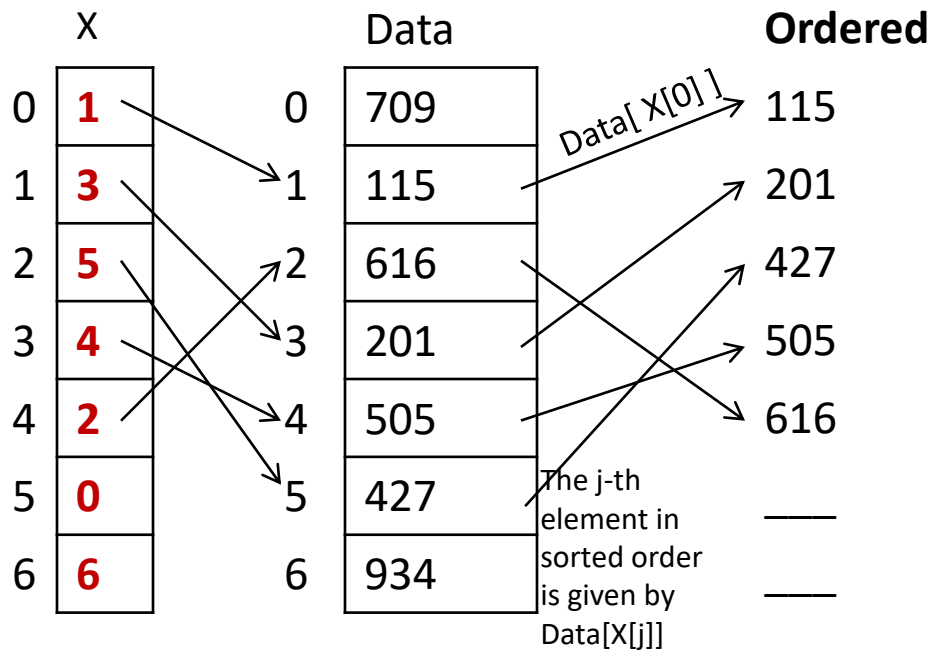



```
1. int search(int A[], int N, int v){
2.    int left, right;
3.    left = 0; right = N-1;
4.    while (left <= right)
5.    { int m = left+(right-left)/2;
6.      if (v == A[m]) return m;
7.      if (v < A[m])
8.          right = m-1;
9.      else
10.         left = m+1;
11.   }
12.   return -1;
13. }
```

use binary search to search for values:
115
950
250

left	right	m	X[m]	Data[X[m]]	Action: Update left/right

Binary Search and Indirect Sorting




```
1. int search(int A[], int N, int v){
2.    int left, right;
3.    left = 0; right = N-1;
4.    while (left <= right)
5.    { int m = left+(right-left)/2;
6.      if (v == A[m]) return m;
7.      if (v < A[m])
8.          right = m-1;
9.      else
10.         left = m+1;
11.   }
12.   return -1;
13. }
```

use binary search to search for values:

115

950

250

left	right	m	X[m]	Data[X[m]]	Action: Update left/right

Binary Search

Binary Search Iterative

- Problem: Determine if object **v** is in array **A**. Assume **A** has size **N** and is sorted in ascending order.
- Reduces the search range in half, with a few instructions.
- See animation: https://www.cs.usfca.edu/~galles/visualization/Search.html
  - The array stretches on 2 or more lines

Search for **392** in sorted array.

v = 392

left	right	middle	Action (comparison)

0	115
1	201
2	427
3	505
4	616
5	709
6	934

```
/* Determines if v is an element of A.
  If yes, returns the position of v in A.
  If not, returns -1. N is the size of A */

1. int binary_search(int A[], int N, int v){
2.    int left, right;
3.    left = 0; right = N-1;
4.    while (left <= right)  {
5.      int m = left+(right-left)/2;
6.      if (v == A[m]) return m;
7.      if (v < A[m])
8.          right = m-1;
9.      else
10.         left = m+1;
11.   }
12.   return -1;
13. }
```

Binary Search

Candidates: N, N/2, N/(2^2) , N/(2^3), …. , , N/(2^x), …., N/(2^p) =1 (last value for which the loop will start) => p = $\log_2 N$ => $\log_2 N$ repetitions
TC_{1iter}() = Θ(1), indep of current number of candiates = right-left+1) =>
Time complexity: Θ($\log_2 N$) (logarithmic. V is compared with at most $\log_2 N$ items.
Space complexity: Θ(1)

Search for v=392 in sorted array:

left	right	middle	Action (comparison)
0	6	3	392 < 505 (go left)
0	2	1	392 > 201 (go right)
2	2	2	392 < 427 (go left)
2	1		Indexes cross, stop. Not found

0	115
1	201
2	427
3	505
4	616
5	709
6	934

```
/* code from Sedgewick
   Determines if v is an element of A.
   If yes, returns the position of v in A.
   If not, returns -1.   N is the size of A.
*/
1. int binary_search(int A[], int N, int v){
2.   int left, right;
3.   left = 0; right = N-1;
4.   while (left <= right) {
5.     int m = left+(right-left)/2;
6.     if (v == A[m]) return m;
7.     if (v < A[m])
8.       right = m-1;
9.     else
10.      left = m+1;
11.  }
12.  return -1;
13. }
```

Binary Search - Recursive

```
/* Adapted from Sedgewick
*/
int binary_search_rec(int A[], int left, int right, int v)
  {
    if (left > right) return -1;

    int m = left+(right-left)/2;
    if (v == A[m]) return m;
    if (v < A[m])
         return binary_search_rec(A, left,  m-1, v);
    else
          return binary_search_rec(A, m+1, right, v);

  }
```

- How many recursive calls?
- See the correspondence between this and the iterative version.

Interpolated search
covered if time permits

- Money winning game:
  - There is an array, A, with 100 items.
  - The items are values in range [1,1000].
  - A is sorted.
  - Values in A are hidden (you cannot see them).
  - You will be given a value, val, to search for in the array and need to either find it (uncover it) or report that it is not there.
  - You start with $5000. For a $500 charge, you can ask the game host to flip (uncover) an item of A at a specific index (chosen by you). You win whatever money you have left after you give the correct answer. You have one free flip.

Value, val, you are searching for.	What index will you flip?	
524		
100		
10		

Index	0	1	…		…	98	99
A							

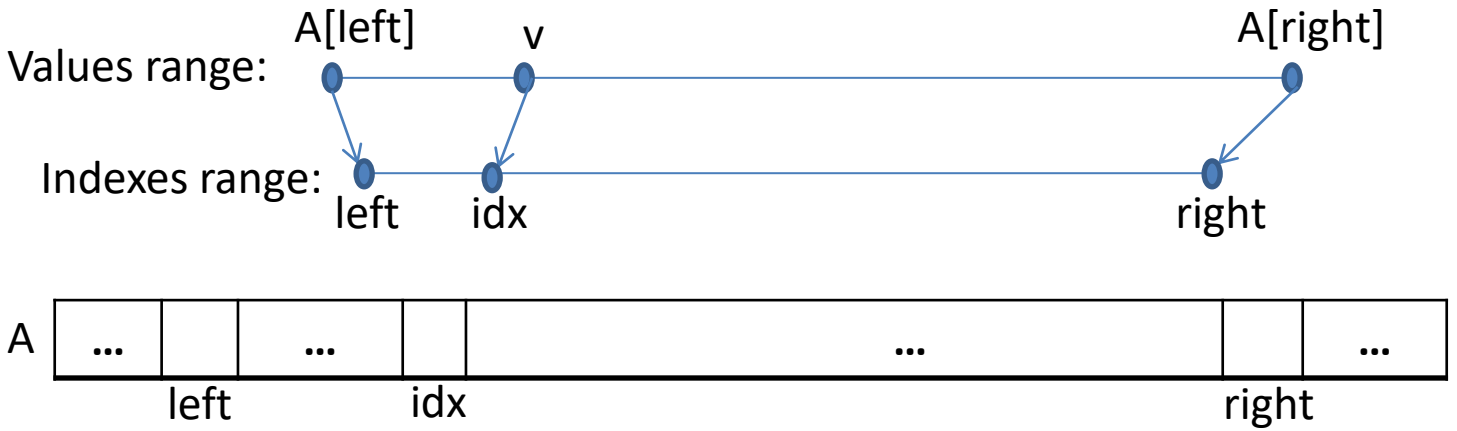
- Money winning game – Version 2 only specific indexes can be flipped.
  - There is an array, A, with 100 items.
  - The items are values in range [1,100].
  - A is sorted.
  - Values in A are hidden (you cannot see them).
  - You will be given a value, val, to search for in the array and need to either find it (uncover it) or report that it is not there.
  - You start with $5000. For a $500 charge, you can ask the game host to flip (uncover) an item of A at a specific index (chosen by you). You win whatever money you have left after you give the correct answer. You have one free flip.

Value, val, you are searching for.	What index will you flip? 0?, 10?, 25?,50?, 75?, 90?, 99?	
524		
100		
10		

Index	0	1	…		…	98	99
A							

Interpolated Binary Search
idx = ??

How will you compute the index *idx* of the best element to inspect?



You want *idx* to be as far away from *left* relative to the indexes range (*right-left*) as *v* is from *A[left]* relative to the values range (*A[right]-A[left]*).

It's all relative!
v = 50
left = 10
right =40

Case 1:
A[left]=100
A[right]=600
idx = ...

Case 2:
A[left]=100
A[right]=140
idx = ……

$$\frac{idx-left}{right-left} = \frac{(v-A[left])}{(A[right]-A[left])} \Rightarrow \boxed{idx = left + \frac{(right-left)}{(A[right]-A[left])}(v - A[left])}$$

Range Transformations (Math review)

- Draw and show the mappings of the interval edges.

- [0,1) -> [0,n)

$$y = xn$$

- [a,b) -> [0,1) -> [0,n)

$$y = \frac{x-a}{b-a} \qquad z = yn$$

- [a,b) -> [0,1) -> [s,t)

$$z = y(t-s)+s$$

if $[a,b) \rightarrow [0,n)$:

$$z = \frac{x-a}{b-a+1}n$$

As a check, see that $a \rightarrow 0$ and $b \rightarrow y < n$.

Direct formula for $[a,b) \rightarrow [s,t)$:

$$z = \frac{x-a}{b-a}(t-s)+s$$

As a check, see that $a \rightarrow s$ and $b \rightarrow t$.

  - What this transformation is doing is: bring to origin (a->0), scale to 1, scale up to new scale and translate to new location s. The order mtters! You will see this in Computer Graphics as well.

Pseudocode

(CLRS Page 20)

Conventions

- Indentation shows body of loop or of a branch
- y = x treated as pointers so changing x will change y.
- cascade: x.f.g
- NILL used for the NULL pointer
- Pass by value of pointer: if x is a parameter, x=y will not be preserved but x.j=3 will be (when returned back to the caller fct)
- Pseudocode will allow multiple values to be returned with one return statement.
- The Boolean operators “and” and “or” are short circuiting: “x != NILL and x.f!=3 ” is safe.
- “the keyword “error” indicates that an error occurred because the conditions were wrong for the procedure to be called.” - CLRS

Questions?

- Note lack of details: no types, no specific syntax (C, Java,…)
- But sufficient specifications to implement it: indexes, data updates, arguments, …