Summations

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Math background - Review

- Series (the terms) and their summations:
  - Geometric
  - Arithmetic
  - (This information is in the following slides and the class cheat sheet)

- Integrals
  - From the Integrals cheat sheet see:
    - The fundamental theorem of calculus Part II:
    - Common integrals

- Cheat sheets and other useful links are on the Slides and Resources webpage.

Book Reference

- Book (CLRS) references for this lecture: Apendix A, especially
  - pages 1145-1147,
  - page 1150 (bounding the terms),
  - Pages 1154-1156 (approximation by integrals)

Overview

- Summations
  - Summation of *arithmetic series:* $\sum_{k=0}^{t}(a_1 + kd)$
    - Where a_1 is the first term and *d* is the step.
  - Summation of *geometric series*: $\sum_{k=0}^{n} x^k$
    - $0 < x < 1$, $\Theta(1)$
    - $x > 1$, $\Theta(x^n)$
    - $x = 1$, $\Theta(n)$ (easy to check: 1+1+…+1)
  - Approximation by integrals
  - Induction
    - You suspect/guess that a summation S = $\Theta\big(f(N)\big)$ and prove/verify it using induction.

    See Dr. Bob Weems, “Notes 3: Summations”, section 3.D.
      - Very useful exercise in strengthening one’s math skills.

Summations

- Summations are formulas of the sort:

$$\sum_{k=0}^{n} f(k)$$

- Used to solve recurrences and time complexity of loops

Summation of Arithmetic Series

- Examples:
  - 1,5,9,13,17,21,….,
    - ‘Step’ between terms: d = 4
    - First term: $a_1 = 1$
  - Note that 1,2,3,…,n is a special case where the step is +1.
  - 3,13,23,33,43,….
    - Step? First term (a_1)?
- General formula: $a_i = a_1 + (i-1)d$
  - Here *d* is the step and a_1 is the first term

- Summation: $\sum_{i=1}^{n} a_i = n\frac{(a_1 + a_n)}{2} = \frac{n}{2}[2a_1 + (n-1)d]$
- Summation of consecutive terms: $1+2+3+\ldots+n = \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$
- Do not confuse the summation of terms with the count of terms! (See a future slide)

Summation of Geometric Series

$0 < x < 1$

- Suppose that $\mathbf{0} < \boldsymbol{x} < \mathbf{1}$:
- Finite summations: $\sum_{k=0}^{n} x^k = \frac{x^{n+1}-1}{x-1} = \frac{1-x^{n+1}}{1-x} = \Theta(??)$
  - Infinite summations: $\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$
  - Important to note: $\sum_{k=0}^{n} x^k \le \sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$
  - Therefore, $\sum_{k=0}^{n} x^k = \Theta(1)$. Why?

Summation of Geometric Series

$0 < x < 1$

- Suppose that $\mathbf{0} < \boldsymbol{x} < \mathbf{1}$:
- Finite summations: $\sum_{k=0}^{n} x^k = \dfrac{1 - x^{n+1}}{1 - x} = \Theta(1)$
  - Infinite summations: $\sum_{k=0}^{\infty} x^k = \dfrac{1}{1 - x}$
  - Important to note: $\sum_{k=0}^{n} x^k \leq \sum_{k=0}^{\infty} x^k = \dfrac{1}{1 - x}$
  - Therefore, $\sum_{k=0}^{n} x^k = \Theta(1)$. Why?
  - Because $\frac{1}{1-x}$ is independent of n.
  - Strictly speaking we showed that it is O(1), but the sum is also ≠ 0 and so it is Ω(1).

Summation of Geometric Series
$x \geq 1$

- $x > 1$ The formula is the same, and can be rewritten as:

$$\sum_{k=0}^{n} x^k = \frac{x^{n+1} - 1}{x - 1}$$

  - Remember this formula!

  - For example:

$$1 + 5 + 5^2 + 5^3 + \ldots + 5^n = \frac{5^{n+1} - 1}{5 - 1} = \Theta(5^n)$$

5 is a constant and so: $5^{n+1} = 5 * 5^n = \Theta(5^n)$ (Exponential!)

- $x = 1$ =>

$$\sum_{k=0}^{n} x^k = \sum_{k=0}^{n} 1 = n + 1 = \theta(n)$$

Approximation by Integrals

$$\int_{m-1}^{n} f(x)dx \leq \sum_{k=m}^{n} f(k) \leq \int_{m}^{n+1} f(x)dx$$

when f(x) is a monotonically increasing function.

In most cases***, our end result will be:

$\sum_{k=1}^{n} f(k) = \Theta(\int_{0}^{n} f(x)dx)$ when f(x) is monotonically increasing

*** For the problems we look at, you can adjust the borders since the differences should result in lower order terms (see the worked-out example). But always be prepared to check the correctness.

Approximation by Integrals

$$\int_{m-1}^{n} f(x)dx \leq \sum_{k=m}^{n} f(k) \leq \int_{m}^{n+1} f(x)dx$$

when f(x) is a monotonically increasing function.

$$\int_{m}^{n+1} f(x)dx \leq \sum_{k=m}^{n} f(k) \leq \int_{m-1}^{n} f(x)dx$$

when f(x) is a monotonically decreasing function.

Informally you can use

$\sum_{k=m}^{n} f(k) = \Theta\left(F(n) - F(m)\right)$ where F is the antiderivative of f

Approximation by Integrals - Theory

$$\int_{m-1}^{n} f(x)dx \leq \sum_{k=m}^{n} f(k) \leq \int_{m}^{n+1} f(x)dx$$

- Suppose that f(x) is a monotonically increasing function:
  - This means that $x \leq y \Rightarrow f(x) \leq f(y)$.
- Then, we can approximate summation $\sum_{k=m}^{n} f(k)$ using integral $\int_{m}^{n+1} f(x)dx$.
- Why? Because $\int_{k-1}^{k} f(x)dx \leq f(k) \leq \int_{k}^{k+1} f(x)dx$.
  - Proof for: $f(k) \leq \int_{k}^{k+1} f(x)dx$

  For every x in the interval $[k, k+1]$, $x \geq k$. Since $f(x)$ is increasing, if $x \geq k$ then $f(x) \geq f(k)$. (Remember that the integral is the sum of $f(x)$ on the interval $[k, k+1]$of size 1.)

  - Similar proof for $\int_{k-1}^{k} f(x)dx \leq f(k)$

Approximation by Integrals - Example

$$\int_{m-1}^{n} f(x)dx \leq \sum_{k=m}^{n} f(k) \leq \int_{m}^{n+1} f(x)dx$$

Problem: approximate T(n) = $\sum_{k=1}^{n} \boldsymbol{k^2}$ using an integral:

Answer:

Note that T(n) is a sum of $\boldsymbol{k^2}$ terms. We will use $f(x) = x^2$ to approximate this summation.

$f(x) = x^2$ is a monotonically increasing function.

O: T(n) = $\sum_{k=1}^{n} k^2 \leq \int_{1}^{n+1} x^2 dx = \frac{(n+1)^3+c-1^3-c}{3} = \frac{n^3+2n^2+2n+1-1}{3} = O(n^3)$

Ω : T(n) = $\sum_{k=1}^{n} k^2 \geq \int_{0}^{n} x^2 dx = \frac{n^3+c-0^3-c}{3} = \frac{n^3}{3} = \Omega(n^3)$

$\Rightarrow f(n) = \Theta(n^3)$

Informal solution: $\sum_{k=1}^{n} f(k)$ *=* Θ*(F(n)-F(1)). =>*

$\sum_{k=1}^{n} k^2 = \Theta(\frac{n^3}{3} + c - (\frac{1^3}{3} + c)) = \Theta(\frac{n^3-1}{3}) = \Theta(n^3)$

Worksheet

1) Find Θ for

$a)\sum_{k=1}^{n} k^{d}$

$b)\sum_{k=1}^{n} k^{-1}$

Hint: pay attention to the borders for b).

2) Given summation: $1 + 2^5 + 3^5 + \ldots + N^5$

Can you solve this in terms of Θ, Ω or O ?

3) The summation we left unsolved before is easy now:

$$\sum_{i=1}^{N} \log_2 i = \sum_{i=1}^{N} \frac{\ln i}{\ln 2} = \frac{1}{\ln 2} \sum_{i=1}^{N} \ln i = \frac{1}{\ln 2} \Theta\big(F(N) - F(1)\big) =$$

$$\Theta\big(N ln(N) + N + c - (1 * \ln(1) + 1 + c)\big) = \Theta\big(N ln(N)\big) =$$

$$\Theta\left(N \frac{\log_2 N}{\log_2 e}\right) = \Theta(N \log_2 N)$$

Extra

- Do not confuse the summation of terms with the count of terms!

- Progression: 1,2,3,…n
- Summation of terms: 1+2+…+n = n(n+1)/2=$\Theta(n^2)$
- Count of terms: 1,2,3,…,n => n terms => $\Theta(n)$

- Progression:1, 2, 4, …, 2^i, …2^n
- Summation: $\sum_{i=0}^{n} 2^i = \frac{2^{n+1}-1}{2-1} = \Theta(2^n)$
- Count of terms: n+1

- Progression:1, 2, 4, …, 2^i, …$2^p \le n$ (last power of 2 less or equal to n)
- Summation: $\sum_{i=0}^{p} 2^i = \frac{2^{p+1}-1}{2-1} = \frac{2n-1}{2-1} = \Theta(n) \quad \text{b.c.}\,(2^p = n)$
- Count of terms: 1+lgn (more precisely: 1+floor(lgn))