Graphs

CSE 3318– Algorithms and Data Structures
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References and Recommended Review

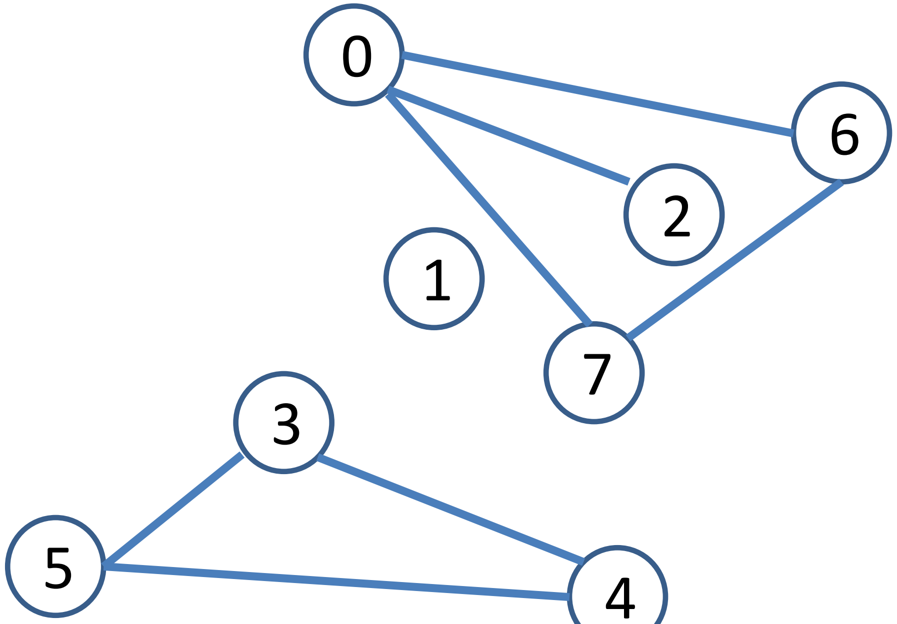
Recommended Student Review from CSE 2315

- Representation
  - Adjacency matrix
  - Adjacency lists
- Concepts:
  - vertex, edge, path, cycle, connected.
- Search:
  - Breadth-first
  - Depth-first

- Recommended: CLRS
- Graph definition and representations
  - CLRS (3rd edition) - Chapter 22.1 (pg 589)
  - Sedgewick - Ch 3.7 (pg 120)
    - 115-120: 2D arrays and lists
- Graph traversal
  - CLRS: BFS - 22.2, DFS-22.3
  - Sedgewick, Ch 5.8
- The code used in slides is from Sedgewick.
- See other links on the Code page.

Graphs

- Graphs are representations of structures, set relations, and states and state-transitions.
  - Direct representation of a real-world structures
    - Networks (roads, computers, social)
  - States and state transitions of a problem.
    - Game-playing algorithms (e.g., Rubik's cube).
    - Problem-solving algorithms (e.g., for automated proofs).
    - For some problems you do not have the entire graph because it is too big. You build it as you go (based on the moves played in the game)
- A graph is defined as **G = (V,E)** where:
  - **V** : set of vertices (or nodes).
  - **E** : set of edges.
    - Each edge is a pair of two vertices in V: $\mathbf{e = (v_1,v_2)}$.

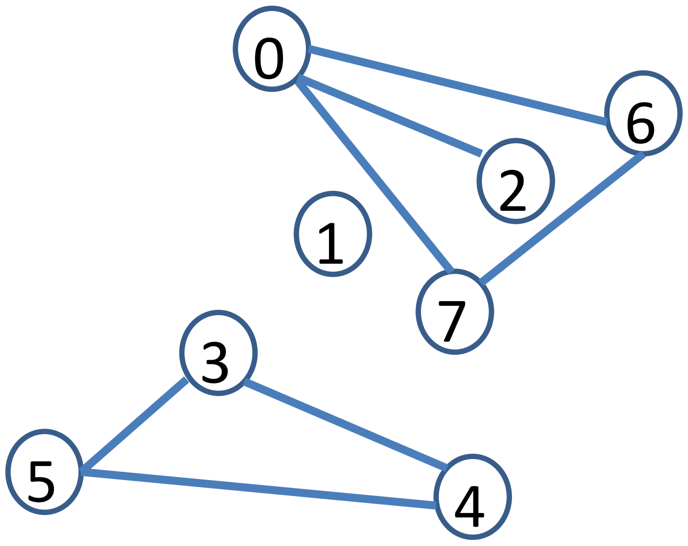


Graphs

- G = (V,E)
  - How many graphs are here?: __1
  - |V| = _8_ , V: { 0, 1, 2, 3, 4, 5, 6, 7 }
  - |E| = _7_ , E: { (0,2), (0,6), (0, 7), (3, 4), (3, 5), (4, 5), (6,7)}.
- Paths
  - Are 2 and 7 connected? Yes: paths 2-0-6-7 or 2-0-7
  - Are 1 and 3 connected? No.
- Cycle
  - A path from a node back to itself.
  - Any cycles here? 3-5-4-3, 0-6-7-0
- Directed / undirected
- Connected component (in undirected graphs)
  - A set of vertices s.t. for any two vertices, u and v, there is a path from u to v.
  - Here: Maximal: {1}, {3,4,5}, {2,0,6,7}. Non-maximal {0,6,7}, {3,5},…
  - *In directed graphs: strongly connected components*.
- Degree of a vertex
  - Number of edges incident to the vertex (for undirected graphs).
  - Here: degree(0) = 3, degree(1) = 0 , degree(5) = 2
- Sparse /dense
- Representation: adjacency matrix, adjacency list

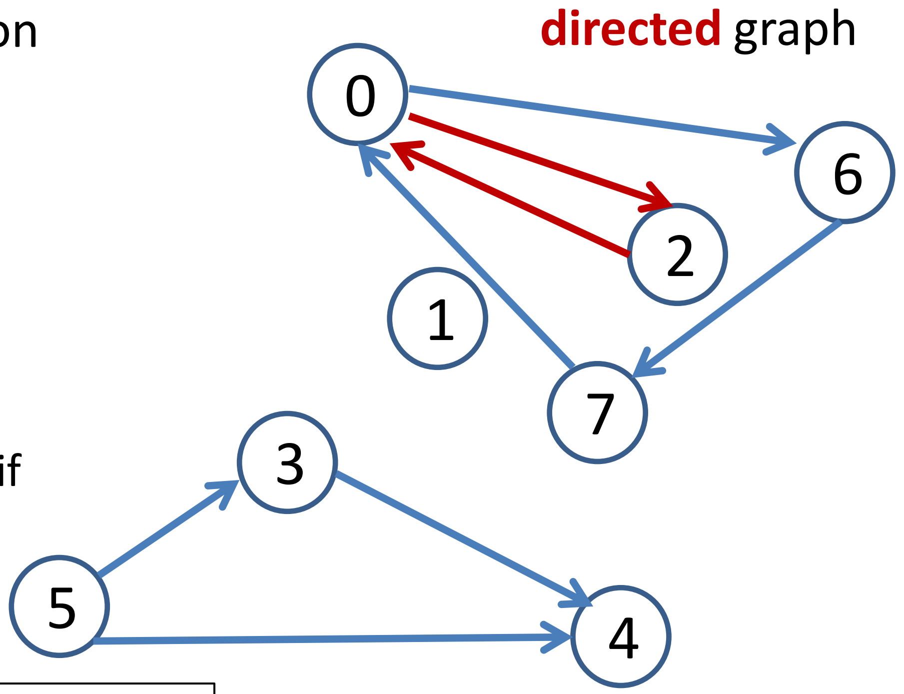
Note: A tree is a graph that is connected and has no cycles

undirected graph



Directed vs Undirected Graphs

- Graphs can be directed or undirected.
- Undirected graph: edges have no direction
  - edge (A, B) means that we can go (on that edge) from **both A to B and B to A.**
- Directed graph: edges have direction
  - edge (A, B) means that we can go (on that edge) **from A to B, but not from B to A.**
  - will have both edge (A, B) and edge (B, A) if A and B are linked in both directions.



Degree of a vertex of a directed graph:
- **In-degree** – number of edges arriving at this vertex
- **Out-degree** – number of edges leaving from this vertex

Vertex	0	4	5	1	7
In degree					
Out-degree					

= E

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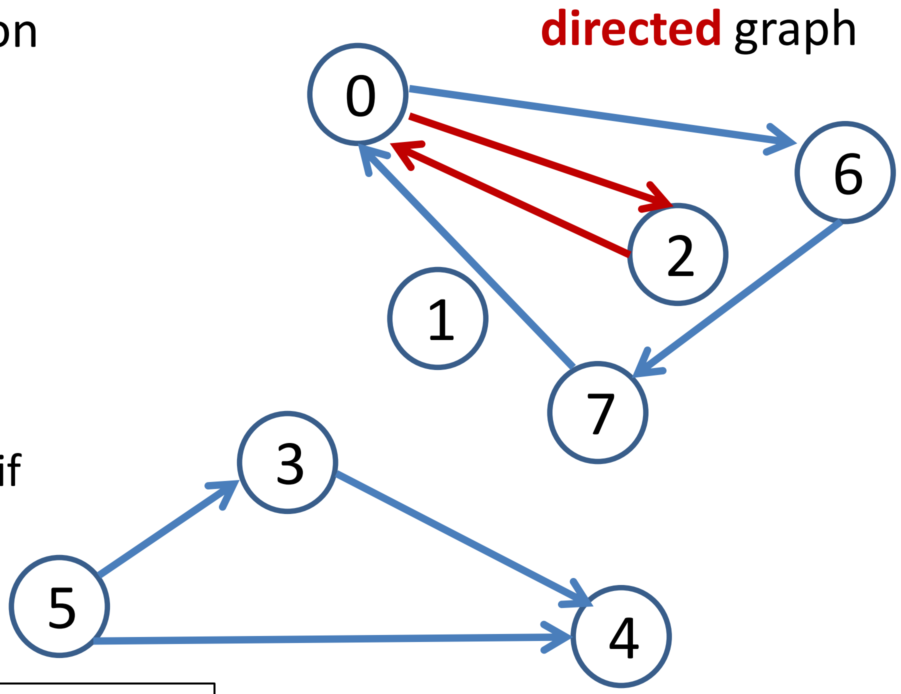


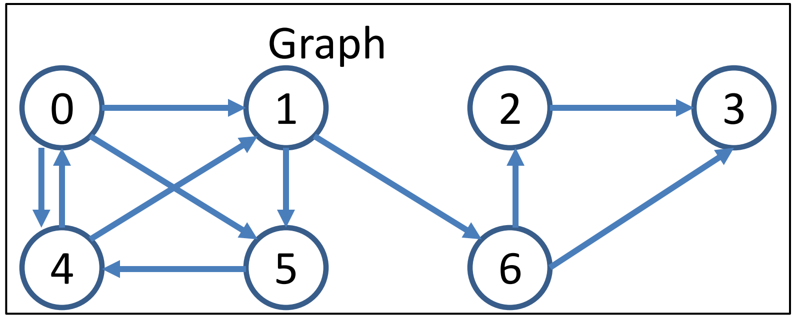
Degree of a vertex of a directed graph:
- **In-degree** – number of edges arriving at this vertex
- **Out-degree** – number of edges leaving from this vertex

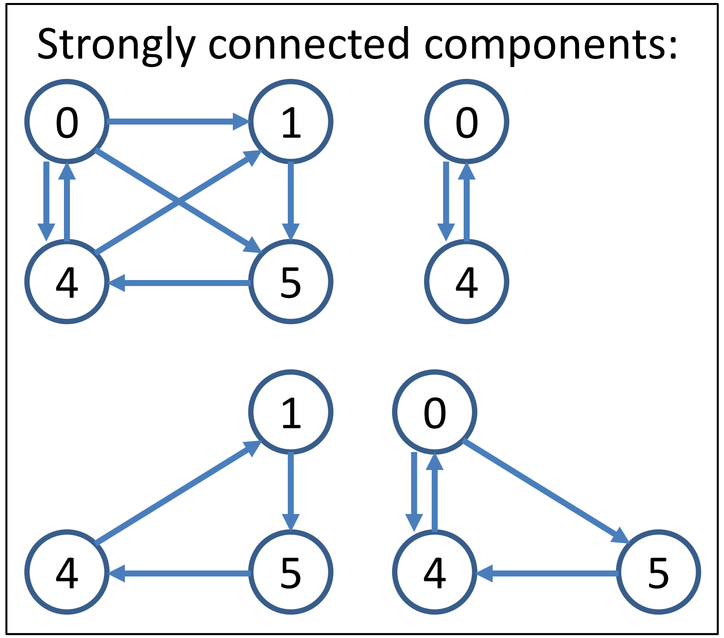
Vertex	0	4	5	1	7
In degree	**2**	**2**	**0**	**0**	**1**
Out-degree	**2**	**0**	**2**	**0**	**1**

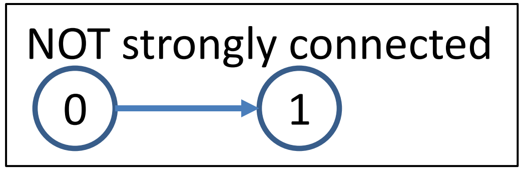
Strongly Connected Components (Directed Graphs)

- How many "connected components" does this graph have?
  1. Can you get from 0 to every other vertex?
  2. Can you get from 3 to 6?

- For directed graphs we define **strongly** connected components: a subset of vertices, V_s, and the edges between them , E_s, such that for any two vertices u,v in V_s we can get from u to v (and from v to u) with only edges from E_s.
  - Strongly connected components in this graph: {0,1,4,5}, {0,4}, {1,5,4}, {0,5,4}
  - NOT strongly connected components. {6,2,3}, {0,1} Why?







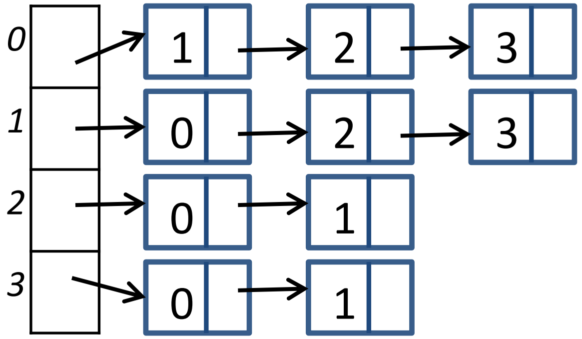
Graph Representations

- G = (V,E). Let |V| = N and |E| = M.
  - |V| is the size of set V, i.e. number of vertices in the graph. Similar for |E|. Notation abuse: V (and E) instead of |V| (and |E|).
- Vertices: store N
  - E.g.: If graph G has N=8 vertices, those vertices will be: 0, 1, 2, 3, 4, 5, 6, 7.
  - Excludes case where additional labels are needed for vertices (e.g. city names).
- Edges: 2 representations:

Adjacency matrix:
A is a 2D matrix of size VxV

	0	*1*	*2*	*3*	*4*	*5*	*6*	*7*
0	0	**1**	**1**	0	0	**1**	**1**	**1**
1	**1**	0	0	0	0	0	0	0
2	**1**	0	0	0	0	0	0	0
3	0	0	0	0	**1**	**1**	0	0
4	0	0	0	**1**	0	**1**	**1**	**1**
5	**1**	0	0	**1**	**1**	0	0	0
6	**1**	0	0	0	**1**	0	0	0
7	**1**	0	0	0	**1**	0	0	0

Adjacency lists:
A is a 1D array of V linked lists



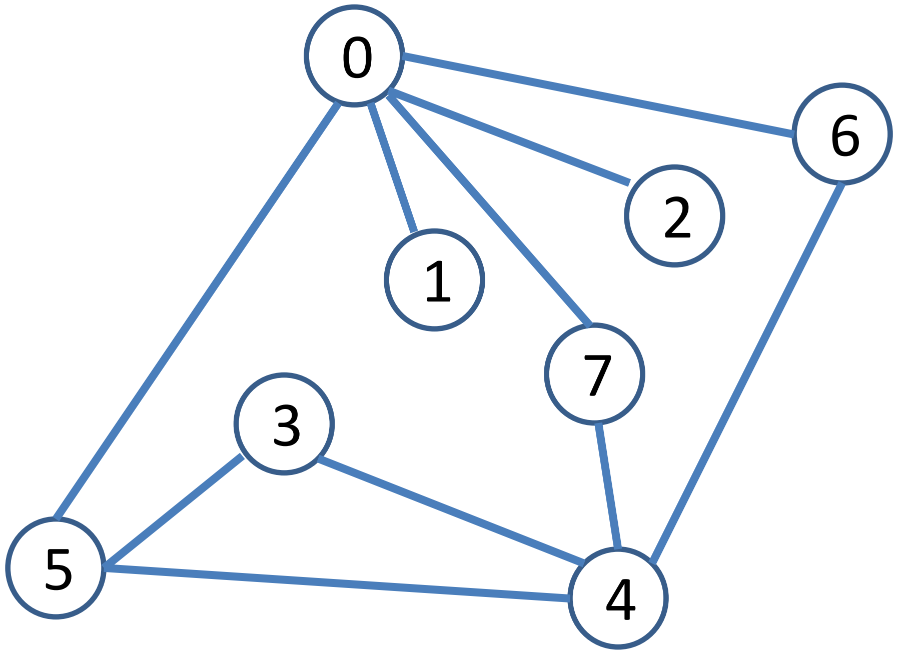
Adjacency Matrix

- Space complexity: $\Theta(V^2)$
- Time complexity for add/remove/check edge: $\Theta(1)$
- Time complexity to find neighbors: $\Theta(V)$

V vertices labelled: 0,1, . . . , V-1.
Represent edges using a 2D matrix, M, of size V*V.

M[x][y] = 1 if and only if there is an edge from x to y.
M[x][y] = 0 otherwise (there is no edge from x to y).

	0	1	2	3	4	5	6	7
0	0	1	1	0	0	1	1	1
1	1	0	0	0	0	0	0	0
2	1	0	0	0	0	0	0	0
3	0	0	0	0	1	1	0	0
4	0	0	0	1	0	1	1	1
5	1	0	0	1	1	0	0	0
6	1	0	0	0	1	0	0	0
7	1	0	0	0	1	0	0	0



Note: the adjacency matrix of non-directed graphs is symmetric.

C implementation for Adjacency Matrix (Undirected graph)

```c
typedef struct struct_graph * graphPT;
struct struct_graph {
    int undirected;
    int V;
    int ** E;
};
graphPT newGraph(int V, int undirected) {
  graphPT res = malloc(sizeof(struct struct_graph));
  res->undirected = undirected;
  res->V = V;
  res->E = alloc_2d(V, V);
// the graph contains no edges (also 0 from caloc).
  for (int i = 0; i < V; i++)
    for (int j = 0; j < V; j++)   res->E[i][j] = 0;
  return res;
}
int edgeExists(graphPT g, int x, int y){   // Θ(1)
    return g->E[x][y];
}
void addEdge(graphPT g, int x, int y){     // Θ(1)
    g->E[x][y] = 1;
    if (g->undirected ==1)     g->E[y][x] = 1;
}
void removeEdge(graphPT g, int x, int y){   //Θ(1)
    g->E[x][y] = 0;
    if (g->undirected ==1)     g->E[y][x] = 0;
}
```

	0	1	2	3	4	5	6	7
0	0	1	1	0	0	1	1	1
1	1	0	0	0	0	0	0	0
2	1	0	0	0	0	0	0	0
3	0	0	0	0	1	1	0	0
4	0	0	0	1	0	1	1	1
5	1	0	0	1	1	0	0	0
6	1	0	0	0	1	0	0	0
7	1	0	0	0	1	0	0	0

```c
void destroyGraph(graphPT g){
    if (g == NULL) return;
    free_2d(g->E, g->V, g->V);
    free(g);
}
```

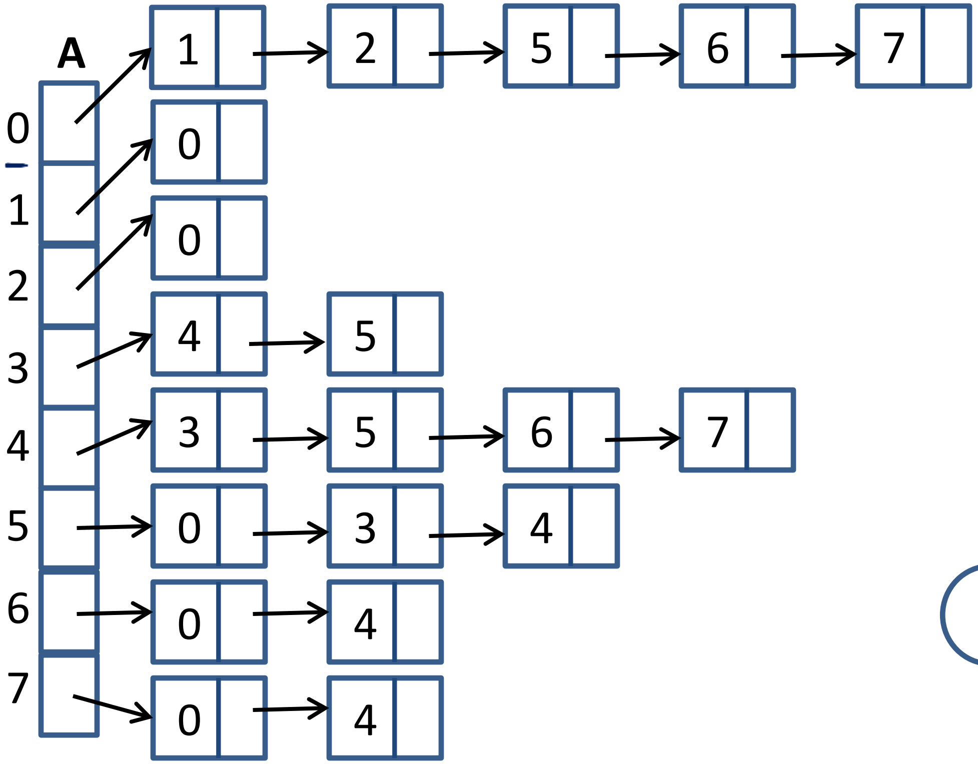
Dynamic 2D array (allocate and free)

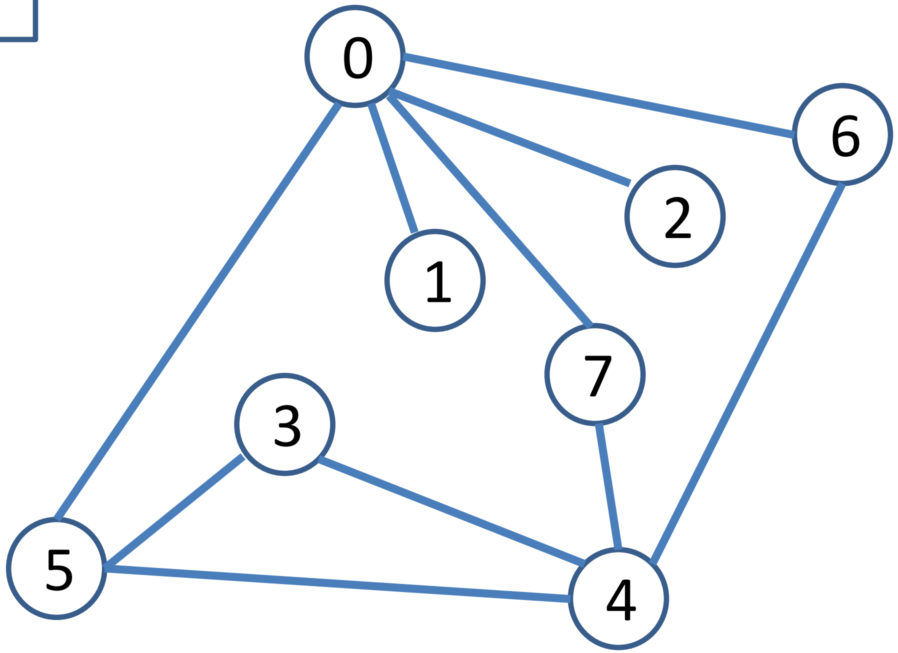
```
// the memory allocated by this function is initialized to 0
int ** alloc_2d(int rows, int columns)
{
   int row;
   // allocate space to keep a pointer for each row
   int ** table = calloc(rows , sizeof(int *));

   // VERY IMPORTANT: allocate the space for each row
   for (row = 0; row < rows; row++)
      table[row] = calloc(columns , sizeof(int));

   return table;
}

void free_2d(int ** array, int rows, int columns)  {

   // VERY IMPORTANT: free the space for each row
   for (int row = 0; row < rows; row++)
      free(array[row]);

   // free the space with the pointer to each row.
   free(array);
}
```

Draw a picture with the data representation.

Adjacency Lists

- Represent the edges of the graph by an **array of linked lists**.
  - Let's name that array A
  - A[x] is a list containing the neighbors of vertex x.

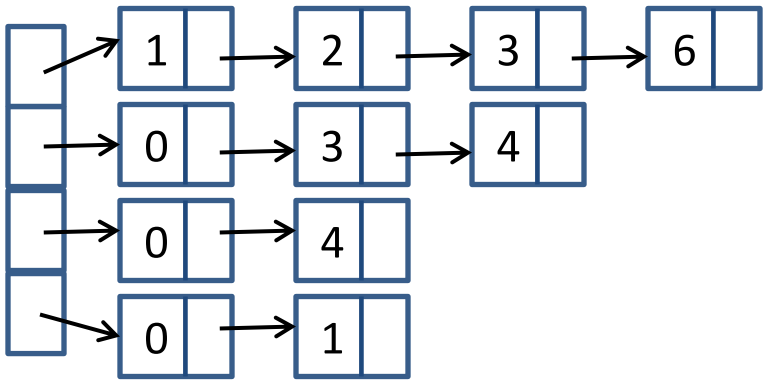




C implementation of Adjacency Lists

```
typedef struct struct_node * nodePT;
struct struct_node{
     int data;
     nodePT next;
}
```

// `struct_graph*` is used to hide the implementation




```
typedef struct struct_graph * graphPT;
struct struct_graph{
   int undirected;
   int V;
   nodePT * E; // array of linked lists
};
```

//Time: $\Theta(deg(x))$, $O(V)$ Space: $\Theta(1)$

```
int edgeExists(graphPT g, int x, int y) {
   for(nodePt n=g->E[x]; n!=NULL; n=n->next)
      if (n->data == y)  return 1;
   return 0;
}
```

//Time: $\Theta(deg(x))$, $O(V)$ Space: $\Theta(1)$

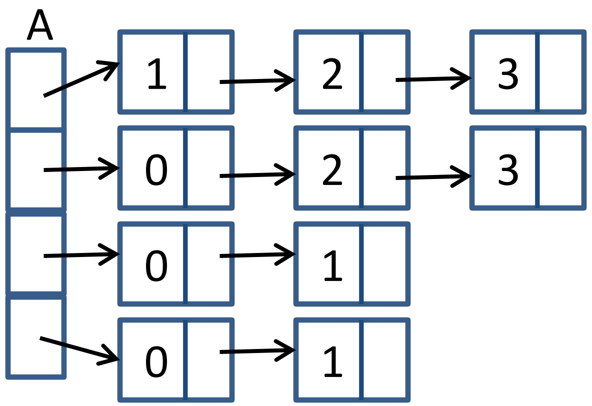
```
void addEdge(graphPT g, int x, int y){
   if (edgeExists(g, x, y)) return;
   g->E[x]=insert_sorted(g->E[x], NULL, new_node(y, NULL));  // insert in order
   if ((x != y) && (g->undirected == 1))
      g->E[y]=insert_sorted(g->E[y], NULL, new_node(x, NULL));  //insert in order
}
```

// Similar for remove edge: iterate through lists of x and y to find the other and remove it.

Adjacency Lists

G(V,E)

- **Space**
  - for A
  - For nodes:



- **Time** to check if an edge exists or not
  - Worst case:

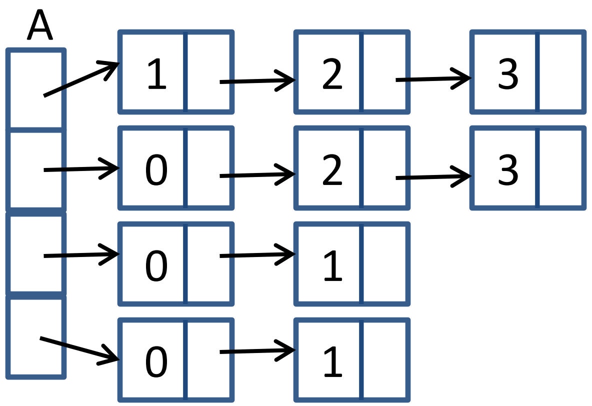
- **Time** to remove an edge?

- **Time** to add an edge?

Adjacency Lists

G(V,E)

- **Space: $\Theta(E + V)$**
  - For A: $\Theta(V)$
  - For nodes: $\Theta(E)$
  - If the graph is relatively sparse, $E << V^2$, this can be a significant advantage.



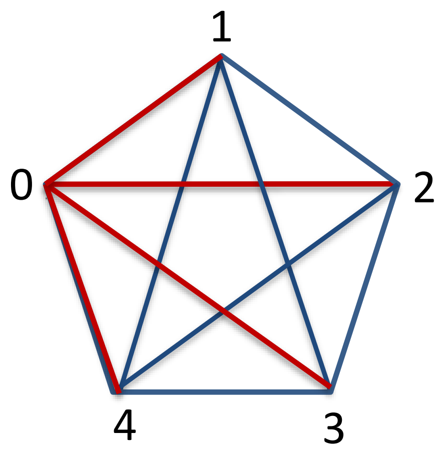
- **Time** to check if an edge exists or not: **O(V)**
  - Worst case: $\Theta(V)$.
    - Each vertex can have up to V-1 neighbors, and we may need to go through all of them to see if an edge exists.
  - Slower than using adjacency matrices.

- **Time** to remove an edge: **O(V)**
  - If must check if the edge exists.
- **Time** to add an edge: **O(V)**
  - If must check if the edge exists.
    - Why? Because if the edge already exists, we should not duplicate it.

Check Out Posted Code

- **graph.h**: defines an abstract interface for basic graph functions.
- **graph_matrix.c**: implements the abstract interface of graph.h, using an adjacency matrix. See also: twoD_arrays.h, twoD_arrays.c for a 2D matrix implemention.
- **graph_list.c**: also implements the abstract interface of graph.h, using adjacency lists.
- **graph_main:** a test program, that can be compiled with **either** graph_matrix.c or graphs_list.c.

Sparse Graphs



- If G(V,E) , max possible edges.
  - Directed: $\Theta(V^2)$ Exact: V*(V-1)
  - Undirected : $\Theta(V^2)$ Exact: [V*(V-1)]/2
- Sparse graph
  - A graph with $E << V^2$ (E much smaller than V^2).
  - https://www.google.com/search?q=image+sparse+graph&tbm=isch&source=univ&sa=X&ved=2ahUKEwiWnLzYpubhAhVSPawKHQ0IDq8QsAR6BAgJEAE&biw=800&bih=528&dpr=2#imgrc=-4yhnsETTHLWcM:
  - E.g. consider an undirected graph with 10^6 nodes
    - Number of edges if 20 edges per node: $10^6*20/2$
    - Max possible edges $10^6*(10^6-1)/2$

    => 10^5 factor between max possible and actual number of edges

    => Use adjacency lists
  - Can you think of real-world data that may be represented as sparse graphs?

Student self study: Space Analysis: Adjacency Matrices vs. Adjacency Lists

- Suppose we have an undirected graph with:
  - 10 million vertices.
  - Each vertex has at most 20 neighbors.

- Individual practice: Calculate the minimum space needed to store this graph in each representation. Use/assume:
  - A matrix of BITS for the matrix representation
  - An int is stored on 8 bytes and a memory address is stored on 8 bytes as well.

  Calculate the space requirement (actual number, not Θ) for each representation.

  Compare your result with the numbers below.

  Check your solution against the posted one. Clarify next lecture any questions you may have.

- Adjacency matrices: we need at least 100 trillion bits of memory, so at least 12.5TB of memory.
- Adjacency lists: in total, they would store at most 200 million nodes. With 16 bytes per node (as an example), this takes at most 3.28 Gigabytes.
- We'll see next how to compute/verify such answers.

Steps for Solving This Problem: understand all terms and numbers

- Suppose we have an undirected graph with:
  - 10 million vertices.
  - Each vertex has at most 20 neighbors.
- Adjacency matrices: we need at least 100 trillion bits of memory, so at least 12.5TB of memory.
- Adjacency lists: in total, they would store at most 100 million nodes. With 16 bytes per node (as an example), this takes 3.28 Gigabytes.

- Find ‘keywords’, understand numbers:
  - 10 million vertices => $10 * 10^6$
  - Trillion = 10^{12}
  - 1 TB (terra bytes) = 10^{12} bytes
  - 1GB = 10^9 bytes
  - 100 Trillion <u>bits</u> vs 12.5 TB (terra <u>bytes</u>)

Solving: Adjacency Matrix

- Suppose we have a graph with:
  - 10 million vertices. => **$V = 10*10^6 = 10^7$**
  - Each vertex has at most 20 neighbors.

- Adjacency matrix representation for the graph:
  - The smallest possible matrix: a 2D array of **bits** =>
  - The matrix size will be: V x V x 1bit =>

  $10^7 * 10^7 * 1\text{bit} = 10^{14}$ bits

  - Bits => bytes:

  1byte = 8bits => 10^{14}bits = $10^{14}/8$ bytes = $100/8*10^{12}$bytes = $12.5*10^{12}$bytes

  - $12.5*10^{12}$bytes = **12.5 TB (final result)**

10^{12}bytes = 1TB

Solving: Adjacency List

- Suppose we have an undirected graph with:
  - 10 million vertices. => **V = 10^7**
  - Each vertex has **at most 20 neighbors**.

- Adjacency lists representation of graphs:
  - For each vertex, keep a list of edges (a list of neighboring vertices)
  - Space for the adjacency list array:

    = 10 million vertices*8 bytes (memory address) = 8*10^7 bytes = 0.08 GB
  - Space for all the nodes (from the list for each vertex):

    ≤ 10^7vertices * (20 neighbors/vertex) = **20*10^7 nodes** = 2*10^8 nodes

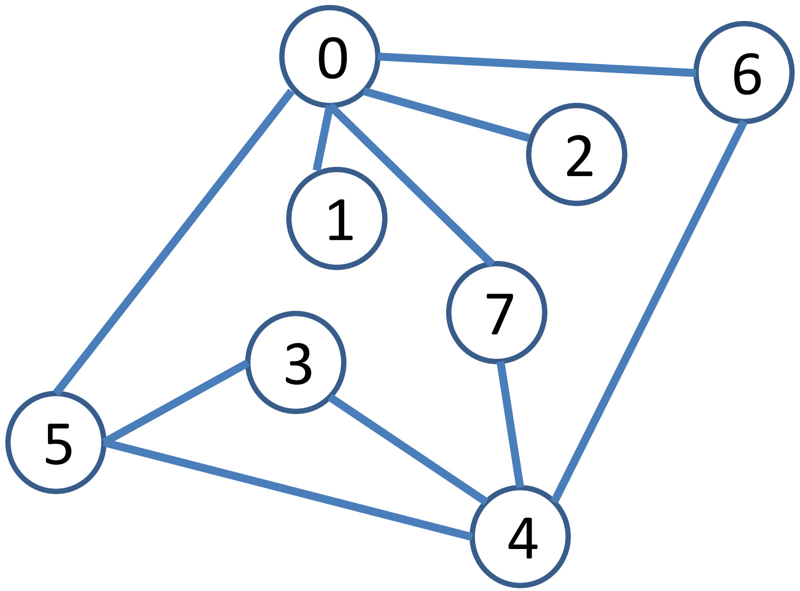
    Assume 16 bytes per node: 8 bytes for the *next* pointer, and 8 bytes for the data (vertex):

    2*10^8 nodes * 16byte/node = 32 * 10^8 bytes = 3.2 * 10^9 bytes = 3.2GB

    Total: **3.2GB + 0.08 GB = 3.28GB** (10^9 bytes = 1GB (GigaByte))

Graph Traversal / Graph Search

- We will use **"graph traversal"** and **"graph search"** almost interchangeably.
  - However, there is a small difference:
    - "Traversal": visit every node in the graph.
    - "Search": visit nodes until find what we are looking for. E.g.:
      - A node labeled "New York".
      - A node containing integer 2014.
- Graph traversal:
  - Input: start/source vertex.
  - Output: a sequence of nodes resulting from graph traversal.



Graph Traversals

Breath-First Search (**BFS**) - call: BFS(G,s)

- O(V+E) (when Adj list repr)

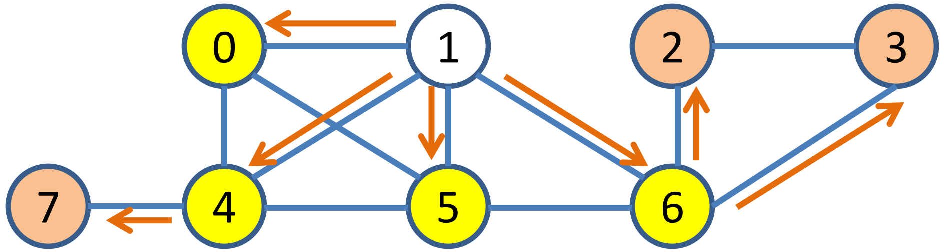
- Explores vertices in the order:
  - root, (white) (Here root = starting vertex, s)
  - vertices 1 edge away from the root, (yellow)
  - vertices 2 edges away from the root, (orange)
  - … and so on until all nodes are visited

- If graph is a tree, gives a level-order traversal.

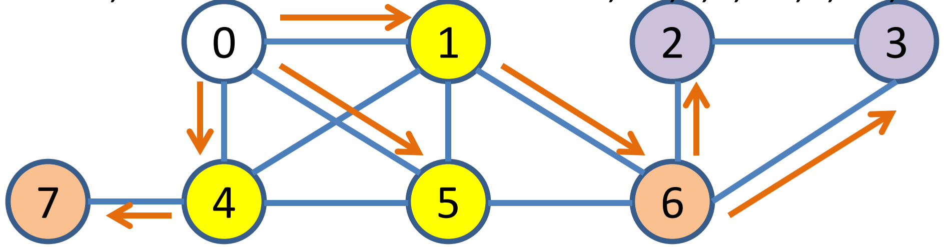
- Finds shortest paths from a source vertex.

*Length of the path is **the number of edges** on it.

E.g. Flight route with fewest connections.



Below, nodes traversed in order: 0, 1,4,5, 6,7, 2,3

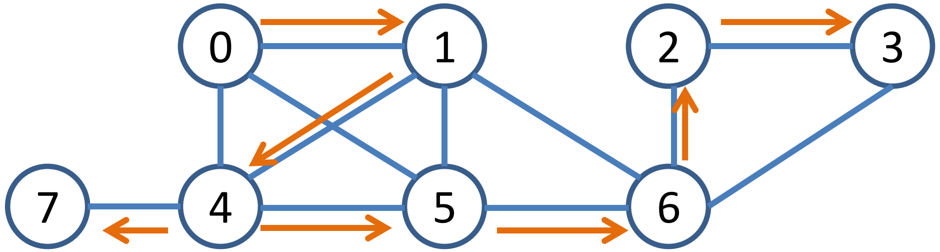


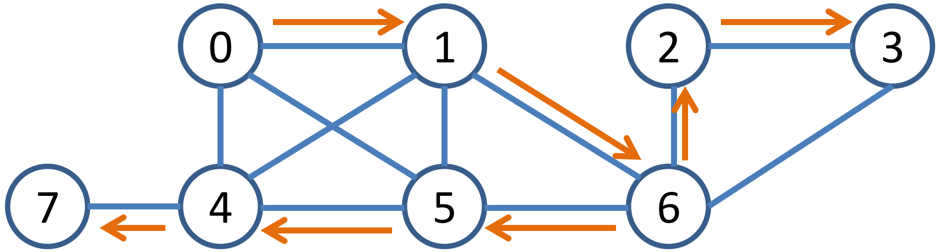
Depth-First Search (**DFS**) - call: DFS(G)

- O(V+E) (when Adj list repr)

- Explores the vertices by following down a path as much as possible, backtracking and continuing from the last discovered node.

- Useful for
  - Finding and labelling strongly connected components (easy to implement)
  - Finding cycles
  - Topological sorting of DAGs (Directed Acyclic Graphs).

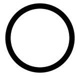




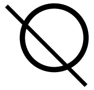
For both DFS and BFS the resulting trees depend on the order in which neighbors are visited.

Vertex coloring while searching

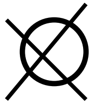
- Vertices will be in one of 3 states while searching and we will assign a color for each state:



  - White – undiscovered



  - Gray – discovered, but the algorithm is not done processing it



  - Black – discovered and the algorithm finished processing it.

Breadth-First Search (BFS)
CLRS 22.2

Space complexity: O(V)

Time complexity:

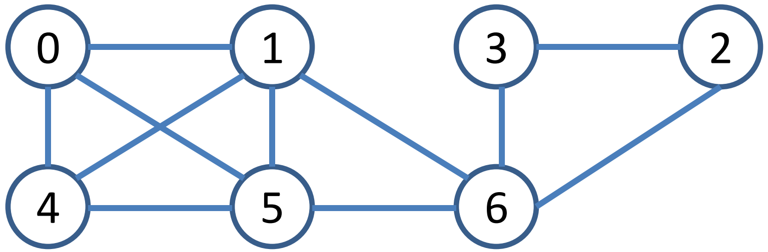
Representation	BFS time complexity
Adj LIST	O(V+E)
Adj MATRIX	$O(V^2)$

BFS-Visit(G,s) *// search graph G starting from vertex s.*

1. For each vertex u of G
   1. color[u] = WHITE *// undiscovered*
   2. dist[u] = inf *// distance from s to u*
   3. pred[u] = NIL *// predecessor of u on the path from s to u*
2. color[s] = GRAY *// s is being processed*
3. dist[s] = 0
4. pred[s] = NIL
5. Initialize empty queue Q
6. put(Q,s) *// s goes to the end of Q*
7. While Q is not empty
   1. u = get(Q) *// removes u from the front of Q*
   2. For each y adjacent to u *//explore edge (u,y) // in increasing order*
      1. If color[y] == WHITE
         1. color[y] = GRAY
         2. dist[y] = dist[u]+1
         3. pred[y] = u
         4. put(Q,y)
   3. color[u] = BLACK

Vertex	Edge	Distance
s		

Queue, Q:



Aggregate time analysis: for each vertex, for each edge => 2*E=>O(E)

Breadth-First Search (BFS)

CLRS 22.2

Solution – to do

Space complexity: O(V)

BFS-Visit(G,s) *// search graph G starting from vertex s.*

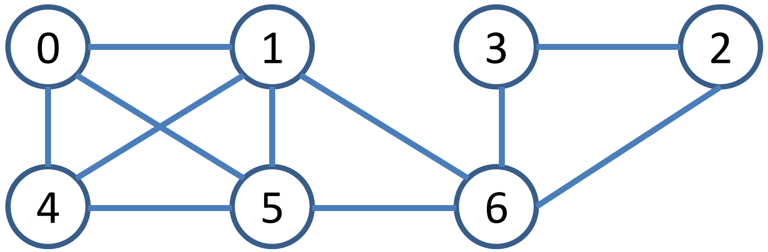
1. For each vertex u of G
   1. color[u] = WHITE *// undiscovered*
   2. dist[u] = inf *// distance from s to u*
   3. pred[u] = NIL *// predecessor of u on the path from s to u*
2. color[s] = GRAY *// s is being processed*
3. dist[s] = 0
4. pred[s] = NIL
5. Initialize empty queue Q
6. put(Q,s) *// s goes to the end of Q*
7. While Q is not empty
   1. u = get(Q) *// removes u from the front of Q*
   2. For each y adjacent to u *//explore edge (u,y) // in increasing order*
      1. If color[y] == WHITE
         1. color[y] = GRAY
         2. dist[y] = dist[u]+1
         3. pred[y] = u
         4. put(Q,y)
   3. color[u] = BLACK

Time complexity:

Representation	BFS time complexity
Adj LIST	O(V+E)
Adj MATRIX	$O(V^2)$

Vertex	Edge	Distance
s		

Queue, Q:



Aggregate time analysis: for each vertex, for each edge => 2*E=>O(E)

Breadth-First Search (BFS):

Note that the code above, CLRS22.2 algorithm, assumes that you will only call BFS(G,s) once for s, and not attempt to find other connected components by calling it again for unvisited nodes.

⇩

If the graph is NOT connected, you will not reach all vertices when starting from s => time complexity is O, not Θ.

(I have seen variation where they restart BFS from the first unvisited node, like DFS)

Depth-First Search (DFS) – simple version

Space complexity: O(____)

Time complexity:

Representation	DFS	DFS-Visit(G,u)
Adj LIST		
Adj MATRIX		

DFS(G)

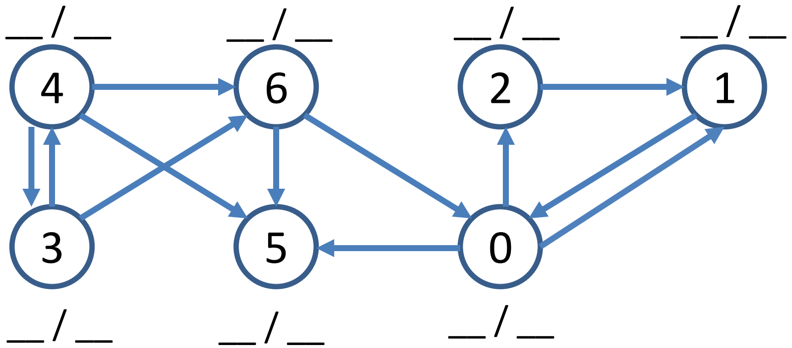
1. For each vertex u of G
   a. color[u] = WHITE
   b. pred[u] = NIL
2. for (u = 0; u<G.V; u++) // for each vertex u of G
   a. If color[u] == WHITE
      1. DFS_visit(G, u, color, pred)

Visited vertex		Pred

List:

DFS_visit(G,u,color, pred)

1. color[u] = GRAY
2. For each y adjacent to u *// explore edge (u,y) // use increasing order for neighbors*
   a. If color[y]==WHITE
      1. pred[y] = u
      2. DFS_visit(G,y, color, pred)
   b. //if color[y]==GRAY then cycle found
3. color[u] = BLACK



Depth-First Search (DFS) – Adj List

Space complexity: O(____)

Time complexity:

Representation	DFS	DFS-Visit(G,u)
Adj LIST		
Adj MATRIX		

Visited vertex		Pred

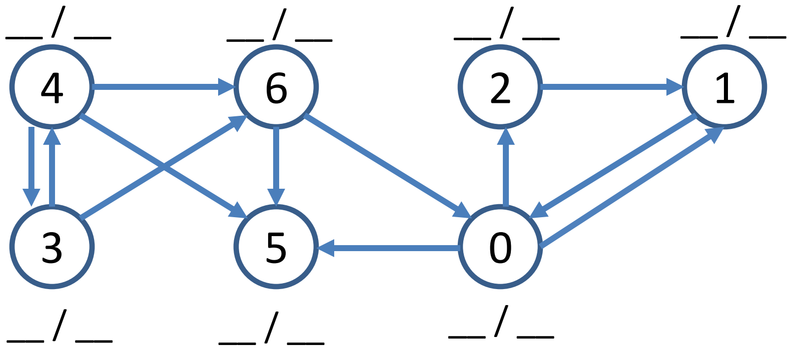
List:

DFS(G)

1. For each vertex u of G
   a. color[u] = WHITE
   b. pred[u] = NIL
2. for (u = 0; u<G.V; u++) // for each vertex u of G
   a. If color[u] == WHITE
      1. DFS_visit(G, u, color, pred)

DFS_visit(G,u,color, pred)

1. color[u] = GRAY
2. ____
   a. If color[y]==WHITE
      1. pred[y] = u
      2. DFS_visit(G,y, color, pred)
   b. //if color[y]==GRAY then cycle found
3. color[u] = BLACK



Depth-First Search (DFS) – Adj Matrix

Space complexity: O(____)

Time complexity:

Representation	DFS	DFS-Visit(G,u)
Adj LIST		
Adj MATRIX		

DFS(G)

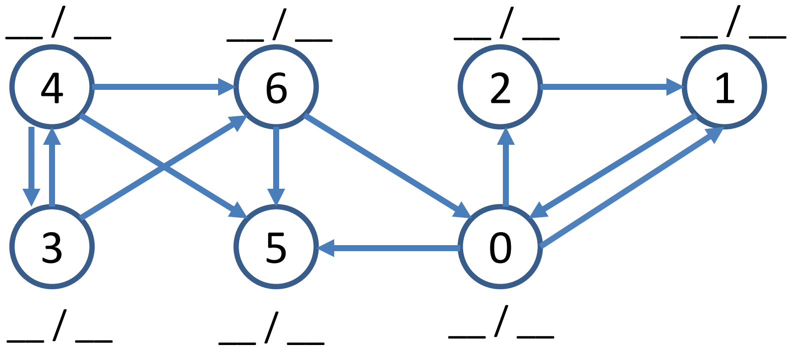
1. For each vertex u of G
   a. color[u] = WHITE
   b. pred[u] = NIL
2. for (u = 0; u<G.V; u++) // for each vertex u of G
   a. If color[u] == WHITE
      1. DFS_visit(G, u, color, pred)

DFS_visit(G,u,color, pred)

1. color[u] = GRAY
2. ____
   a. If color[y]==WHITE
      1. pred[y] = u
      2. DFS_visit(G,y, color, pred)
   b. //if color[y]==GRAY then cycle found
3. color[u] = BLACK

Visited vertex		Pred

List:



Depth-First Search (DFS) – simple version

Space complexity: O(V)

Time complexity:

Representation	DFS	DFS-Visit(G,u)
Adj LIST	$\Theta(V+E)$	Θ(neighbors of u)
Adj MATRIX	$\Theta(V^2)$	$\Theta(V)$

DFS(G)

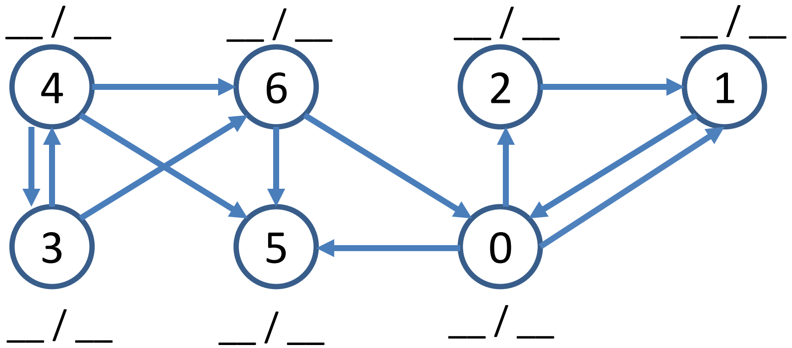
1. For each vertex u of G
   a. color[u] = WHITE
   b. pred[u] = NIL
2. for (u = 0; u<G.V; u++) // for each vertex u of G
   a. If color[u] == WHITE
      1. DFS_visit(G, u, color, pred)

DFS_visit(G,u,color, pred)

1. color[u] = GRAY
2. For each y adjacent to u *// explore edge (u,y) // use increasing order for neighbors*
   a. If color[y]==WHITE
      1. pred[y] = u
      2. DFS_visit(G,y, color, pred)
   b. //if color[y]==GRAY then cycle found
3. color[u] = BLACK

Visited vertex		Pred

List:



Depth-First Search (DFS) – simple version

Space complexity: O(V)

Time complexity:

Representation	DFS	DFS-Visit(G,u)
Adj LIST	$\Theta(V+E)$	Θ(neighbors of u)
Adj MATRIX	$\Theta(V^2)$	$\Theta(V)$

DFS(G)

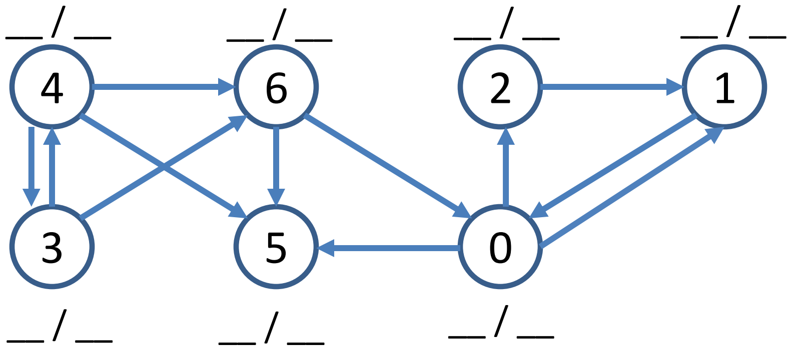
1. For each vertex u of G
   a. color[u] = WHITE
   b. pred[u] = NIL
2. for (u = 0; u<G.V; u++) // for each vertex u of G
   a. If color[u] == WHITE
      1. DFS_visit(G, u, color, pred)

DFS_visit(G,u,color, pred)

1. color[u] = GRAY
2. For each y adjacent to u *// explore edge (u,y) // use increasing order for neighbors*
   a. If color[y]==WHITE
      1. pred[y] = u
      2. DFS_visit(G,y, color, pred)
   b. //if color[y]==GRAY then cycle found
3. color[u] = BLACK

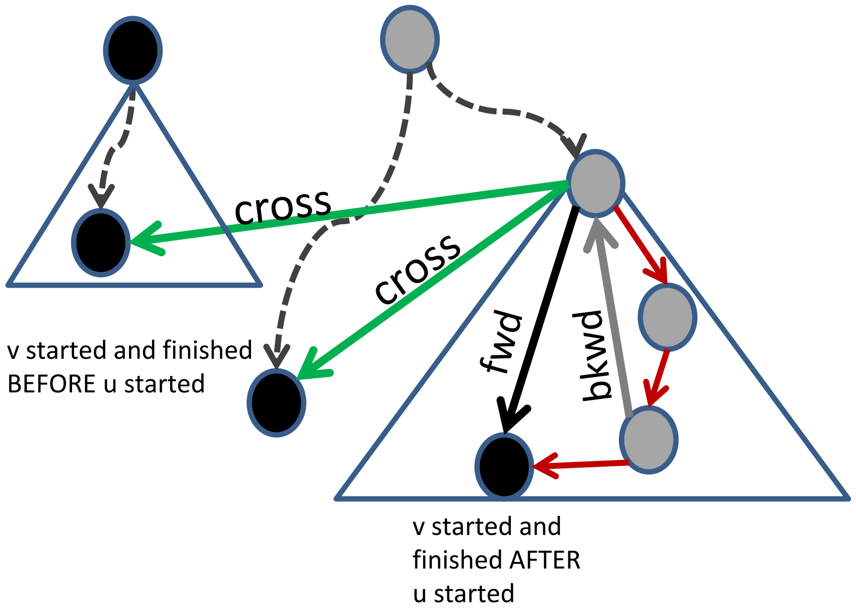
Visited vertex		Pred

List:



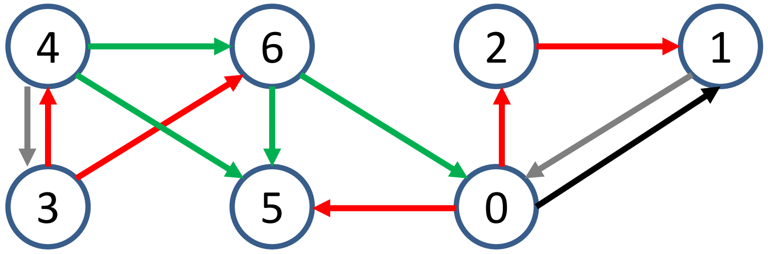
Edge Classification: tree, backward, C/F

Edge type for (u,v)	Color of v	Arrow color (my convention)	Comments
Tree	***White***		***v is first discovered***
Backward	***Gray***		***There is a cycle***
C/F (Forward)	Black		Shortcut. v is a descendant of u v started after u started
C/F (Cross)	Black		v is not a descendant of u v started before u started

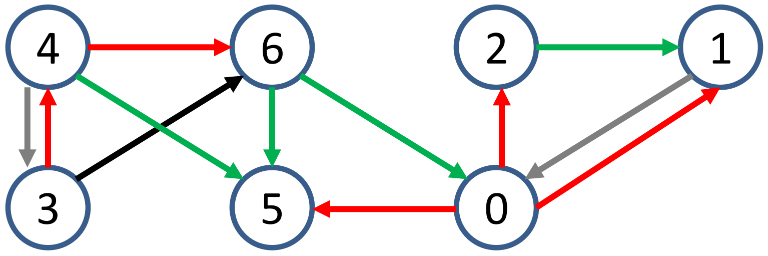


We will use the labels: ***tree, backward, C/F*** and not care if *C/F* is a *forward* or a *cross*.

The edge classification depends on the order in which vertices are discovered, which depends on the order by which neighbors are visited.



DFS(G,0): 0, 2, 1, 5, 3, 6, 4
Visit neighbors of 3 in order: 6 and then 4



DFS(G,0): 0, 1, 2, 5, 3, 4, 6
Visit neighbors in increasing order.

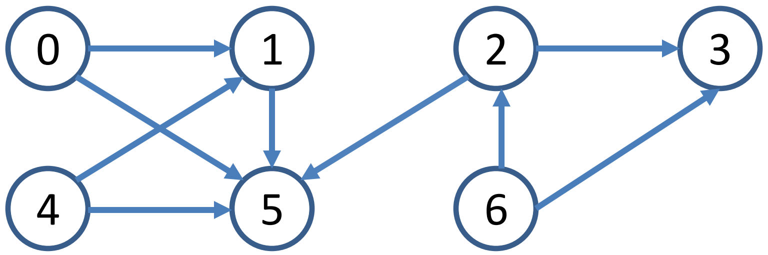
Edge Classification for Undirected Graphs

- An undirected graph will only have:
  - Tree edges
  - Back edges

Topological Sorting

Topological sort of a directed **acyclic** graph (DAG), G, is a linear ordering of its vertices s.t. if (u,v) is an edge in G, then u will be listed before v (all edges point from left to right).

- If a graph has a cycle, it CANNOT have a topological sorting.



Application:

1. Identify strongly connected components in directed graphs.
2. Task ordering (e.g. for an assembly line)
   - Vertices represent tasks
   - Edge (u,v) indicates that task u must finish before task v starts.
   - Topological sorting gives a feasible order for completing the tasks.

Algorithm version 1 (Alexandra)

1. Initialize an array, *res*

2. Run DFS

 - If cycle found, quit => NO topological order

 - Every time a vertex finishes, add it in *res* at next position.

3. Reverse the array *res* and return it. (It will have the vertices listed in decreasing order of DFS finish time).

Algorithm version 2 (CLRS):

1. Initialize an empty linked list L.

2. Call DFS(G) with modification:

When a vertex finishes (black) add it at the beginning of linked list L.

NOTE: If a cycle is detected (backward edge), return null. => No topological order.

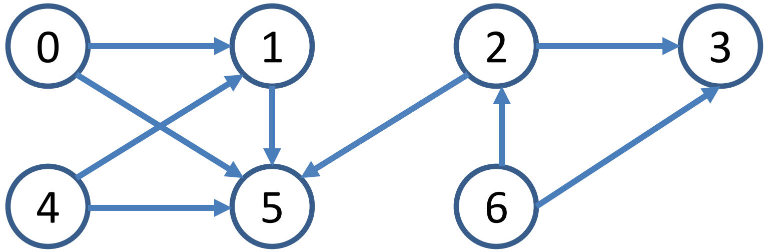
3. Return L

What would you use in Java for L?

Give TC for each version.

Directed Acyclic Graphs (DAG)
&
Detecting Cycles in a Graph

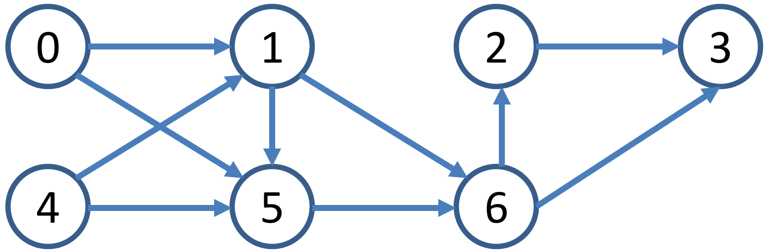
- *A graph has a cycle if a DFS traversal finds a backward edge (an edge that points to a gray node).*
  - Applies to both directed and undirected graphs.
- A Directed Acyclic Graph (DAG) is a directed graph that has no cycles.



Topological Sorting - Worksheet

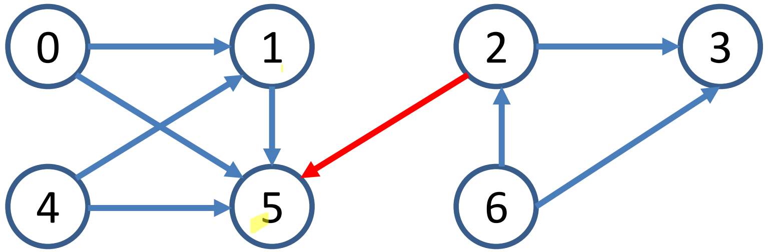
- There may be more than one topological order for a DAG.
  In that case, any one of those is good.
- Red arrows show what is different from the graph in Example 1.

Example 1



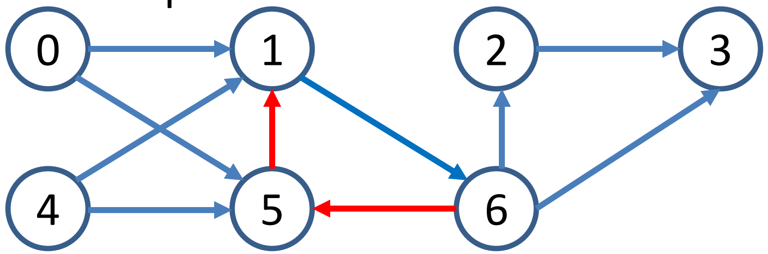
Topological order:

Example 2 (from previous page)



Topological order:

Example 3

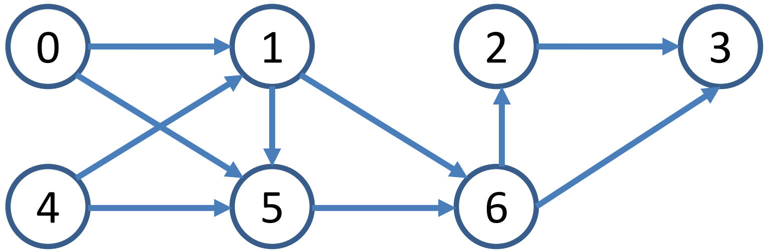


Topological order:

Topological Sorting - Answer

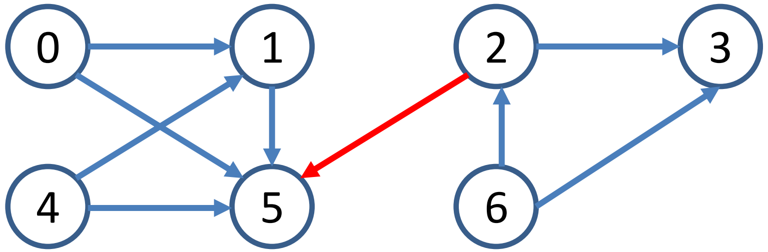
- There may be more than one topological order for a DAG.
  In that case, any one of those is good.
- Red arrows show what is different from the graph in Example 1.

Example 1



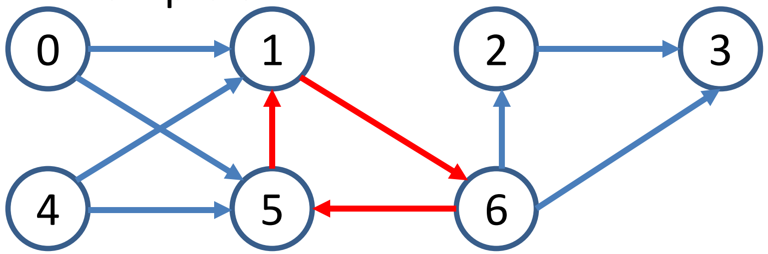
Topological order:
4, 0, 1, 5, 6, 2, 3
(0, 4, 1, 5, 6, 2, 3)

Example 2



Topological order:
6, 4, 2, 3, 0, 1, 5
(0, 4, 1, 6, 2, 3, 5)
(0, 4, 1, 6, 2, 5, 3)

Example 3



Topological order: none. It has a cycle.

Simple pseudocode:

Run DFS and return time finish time data.

- If cycle found, quit => NO topological order

Return array with vertices in reversed order of finish time.

Strongly Connected Components in a Directed Graph

Strongly_Connected_Components(G)

1. `finish1`= DFS(G) //Call DFS and return the vertex finish time, `finish1`

2. Compute G^T

3. Call DFS(G^T), but in its main loop consider the vertices **in order of decreasing finish time,** `finish1`, (i.e. in topological order).

4. Output the vertices of each tree from line 3 as a separate strongly connected component.

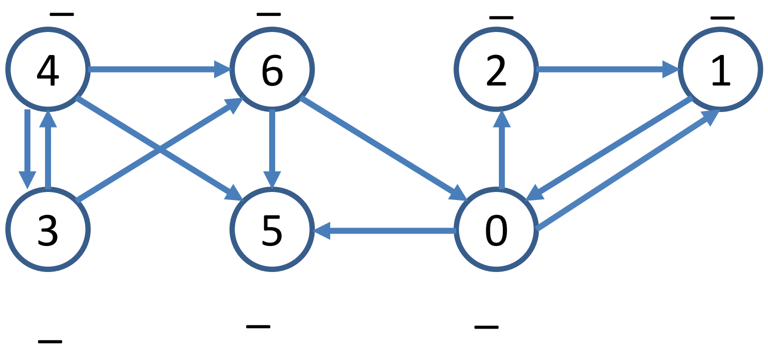
Where: $G^T = (V, E^T)$, with $E^T = \{(y,x) : (x,y) \in E\}$

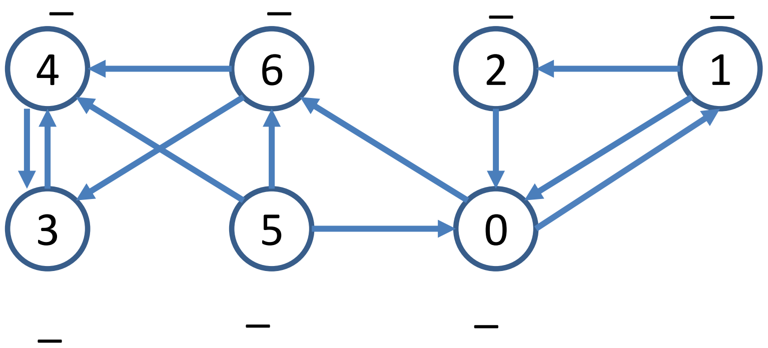
- the *transpose of G*: a graph with the same vertices as G, but with edges in reverse direction.

Visited vertex	Pred	Finish

Finished (reverse order):

(Every node that finishes is added at the front) Consider different list implementations (array/linked list), and the time complexity for them. Can you use a regular array of size N? Can you add every node that finishes at the END?



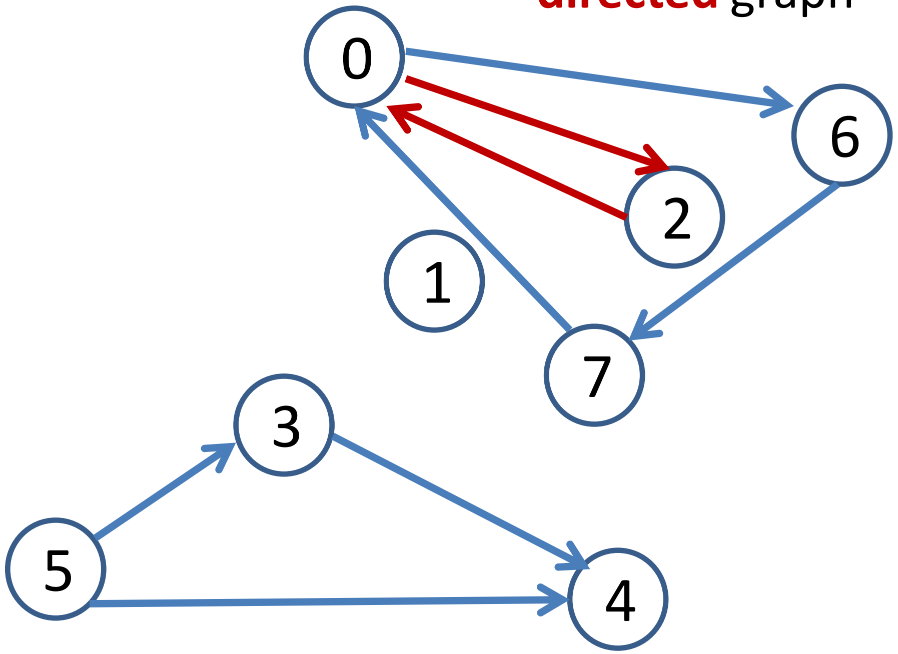


Applications of Strongly Connected Components (SCC)

- Simplify graph: collapse every SCC in one node
- From stackoverflow (https://stackoverflow.com/questions/11212676/what-are-strongly-connected-components-used-for)
  - Model checking - “model checking is applied widely in the industry - especially for proving correctness of hardware components.”
  - Vehicle routing applications – “A road network can be modeled as a directed graph, with vertices being intersections, and arcs being directed road segments or individual lanes. If the graph isn't Strongly Connected, then vehicles can get trapped in a certain part of the graph (i.e. they can get in, but not get out).”
- From Wikipedia (https://en.wikipedia.org/wiki/Strongly_connected_component)
  - SCC “may be used to solve 2-satisfiability problems (systems of Boolean variables with constraints on the values of pairs of variables)”
- From neo4j (https://neo4j.com/docs/graph-algorithms/current/labs-algorithms/strongly-connected-components/)
  - “In the analysis of powerful transnational corporations, SCC can be used to find the set of firms in which every member owns directly and/or indirectly owns shares in every other member. Although it has benefits, such as reducing transaction costs and increasing trust, this type of structure can weaken market competition.”
  - “SCC can be used to compute the connectivity of different network configurations when measuring routing performance in multi hop wireless networks”
- “applications in cell methods for the numerical study of discrete dynamical systems” https://math.stackexchange.com/questions/32041/uses-of-strongly-connected-components

Directed Graphs – Adjacency Matrix - Worksheet

directed graph



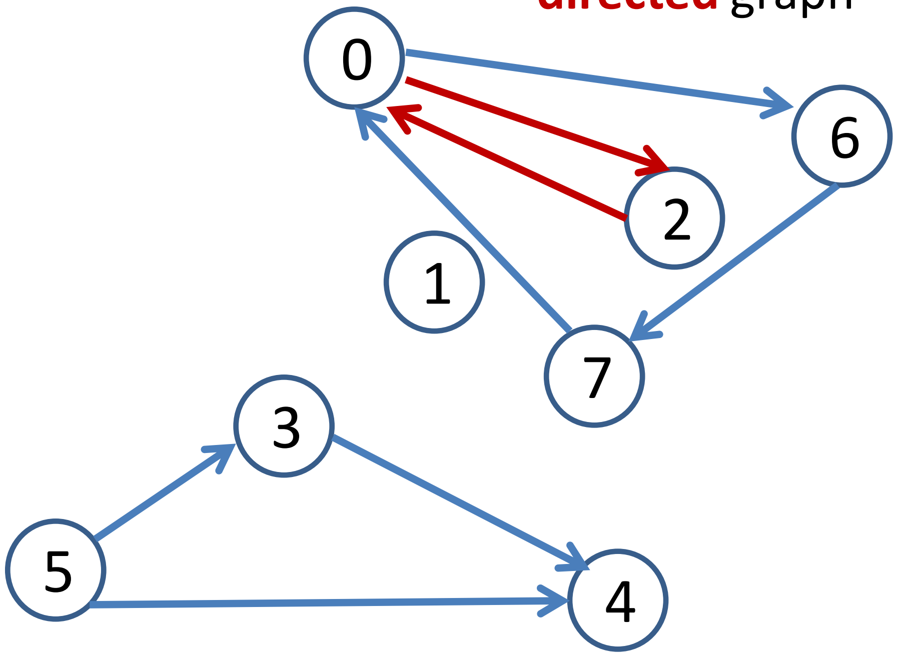
Fill in the matrix representation.
Use row as source and column as destination for edges.
Update it for each edge:
(0,2), (0,6),
(3,4)
(7,0)
…

Degree of a vertex of a directed graph:
- **In-degree** – number of edges arriving at this vertex
- **Out-degree** – number of edges leaving from this vertex

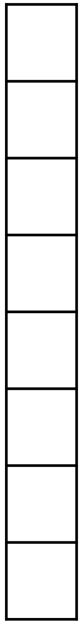
Vertex	0	4	5	1	7
In degree					
Out-degree					

Directed Graphs – Adjacency List - Worksheet

directed graph



Give the Adjacency list representation.
Size __ array of _____________ (type of data in array)
Update it for each edge:
(0,2), (0,6),
(3,4)
(7,1)
…



Degree of a vertex of a directed graph:
- **In-degree** – number of edges arriving at this vertex
- **Out-degree** – number of edges leaving from this vertex

Vertex	0	4	5	1	7
In degree					
Out-degree					

Extra Slides

C implementation for Adjacency Matrix –Undirected graph

```
void graphCreateAndWork() {
  int N;
  scanf("%d",&N);
  int E[N][N];
  for (int i = 0; i < N; i++)
    for (int j = 0; j < N; j++)  E[i][j] = 0;
  // call graph function here, e.g.:
  // addEdge(N,E,1,3);
}
int edgeExists(int N, int E[][N], int v1, int v2){
   if (v1>=N || v1<0 || v2>=N || v2<0)  return -1;
   return E[v1][v2];
}
void addEdge(int N, int E[][N], int v1, int v2){
   if (v1>=N || v1<0 || v2>=N || v2<0)  return;
   E[v1][v2] = 1;
   E[v2][v1] = 1;
}
void removeEdge(int N, int E[][N], int v1, int v2){
   if (v1>=N || v1<0 || v2>=N || v2<0)  return;
   E[v1][v2] = 0;
   E[v2][v1] = 0;
}
```

Simple version where the graph is represented by only N and E.

	0	1	2	3	4	5	6	7
0	0	1	1	0	0	1	1	1
1	1	0	0	0	0	0	0	0
2	1	0	0	0	0	0	0	0
3	0	0	0	0	1	1	0	0
4	0	0	0	1	0	1	1	1
5	1	0	0	1	1	0	0	0
6	1	0	0	0	1	0	0	0
7	1	0	0	0	1	0	0	0

Remember that you cannot return, from a function, an array allocated on the stack (with [][]) . Function graphCreateAndWork *must NOT* return E.

DFS with time stamps

- Next, DFS with ‘time stamps’ of when a node *u* was first discovered (*d[u]*) and the time when the algorithm finished processing that node (*finish[u]*).
- The time stamps are needed for:
  - Topological sorting
  - Finding strongly connected components
  - edge labeling (to distinguish between forward and cross edges)
    - tree edge
    - backward edge
    - forward edge
    - cross edge
- The following pseudo-code does not specify all the details of the implementation

Depth-First Search (DFS)

(with time stamps) - CLRS

DFS(G)

1. For each vertex u of G
   1. color[u] = WHITE
   2. pred[u] = NIL
   3. d[u] = -1
   4. finish[u] = -1
2. *time* = 0
3. For (u=0;u<G.N;u++) *//each vertex u of G*
   1. If color[u] == WHITE *// u is in undiscovered*
      1. **DFS_visit(G, u ,&time, color, pred, d, finish)**

// Search graph G starting from vertex u. u must be WHITE

DFS_visit(G,int u, int* time, ColType*color, int* pred, int* d, int* finish)

1. *(*time) = (*time)* + 1
2. d[u] = *time* *// time when u was discovered*
3. color[u] = GRAY
4. For each v adjacent to u *// assume increasing order*
   1. If color[v]==WHITE *// if color[v]==GRAY=>cycle found*
      1. pred[v] = u
      2. DFS_visit(G,v,…..)
5. color[u] = BLACK
6. *(*time) = (*time)* + 1 *//no two time stamps are equal.*
7. finish[u] = time (See CLRS page 605 for step-by-step example.)

When coding you can use any convention to represent the colors: strings, chars(w/b/g), int (0,1,2), etc.

In the graph picture below, assume no answer means the initial values: NIL, -1,-1

Node u will be:
- WHITE before time d[u],
- GRAY between d[u] and finish[u],
- BLACK after finish[u]

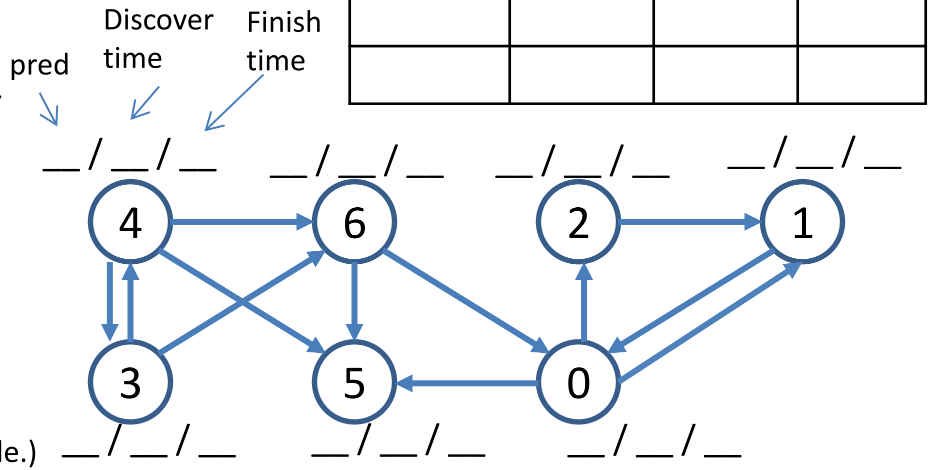
WHITE | GRAY | BLACK
d[u] finish[u]

Time complexity:

Representation	DFS	DFS-Visit(G,u)
Adj LIST	$\Theta(\|V\|+\|E\|)$	Θ(neighbors of u)
Adj MATRIX	$\Theta(\|V\|^2)$	$\Theta(\|V\|)$

Space complexity: O(V)

Visited vertex	Pred	Start	Finish

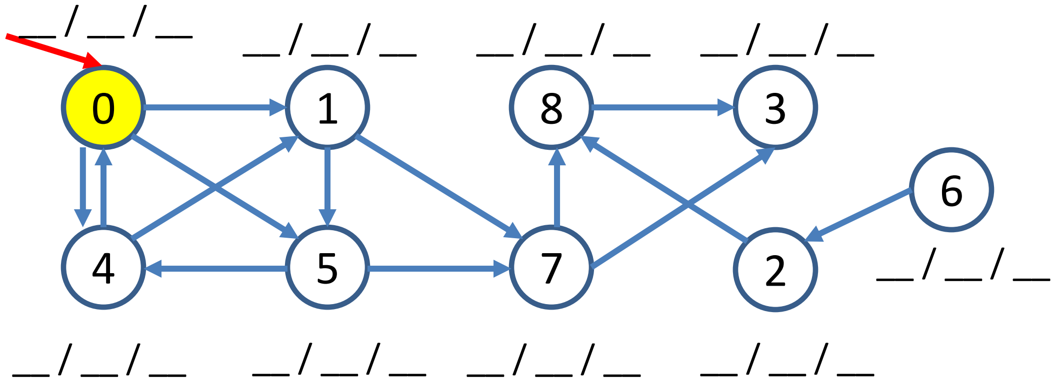


Worksheet

Convention:
start /end
pred

Run DFS on the graphs below. Visit neighbors of u in increasing order.
For each node, write the start and finish times and the predecessor.
Do edge classification as well.

Order from 1st to last	Visited vertex	Pred	Start	Finish

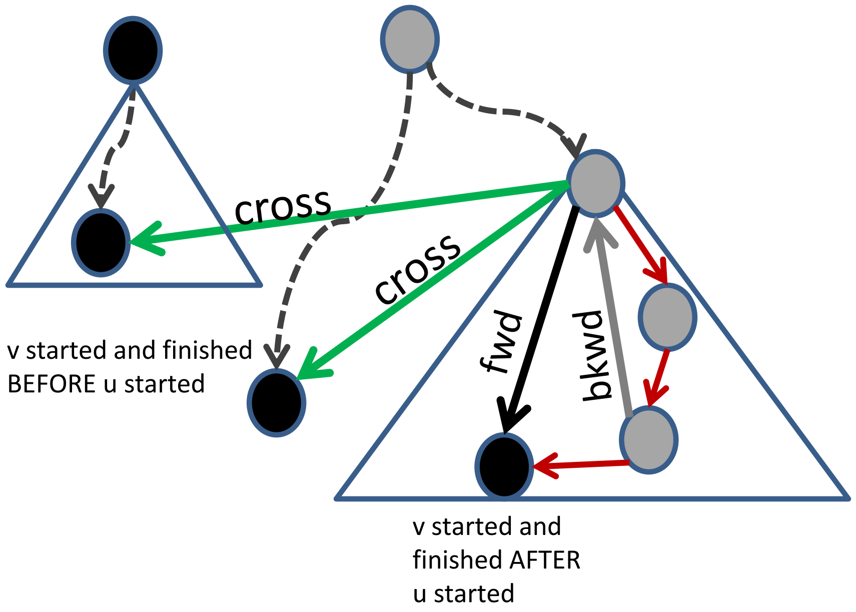


DFS(G) (note that 0 moved)

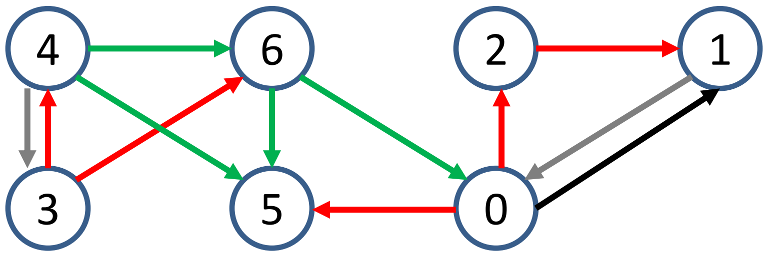
DFS(G):

Edge Classification

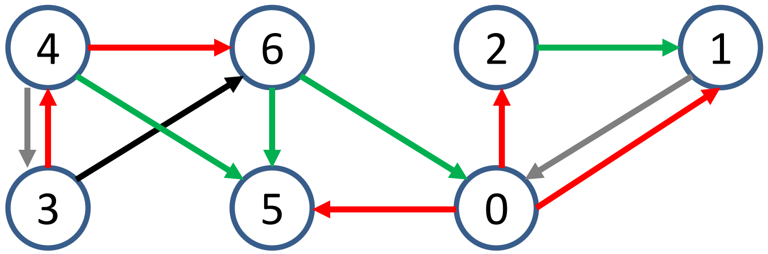
Edge type for (u,v)	Color of v	Arrow color (my convention)	Comments
Tree	White		***v is first discovered***
Backward	Gray		***There is a cycle***
Forward	Black		Shortcut. v is a descendant of u v started after u started
Cross	Black		v is not a descendant of u v started before u started



The edge classification depends on the order in which vertices are discovered, which depends on the order by which neighbors are visited.



DFS(G,0): 0, 2, 1, 5, 3, 6, 4
Visit neighbors of 3 in order: 6 and then 4



DFS(G,0): 0, 1, 2, 5, 3, 4, 6
Visit neighbors in increasing order.

Edge Classification for Undirected Graphs

- An undirected graph will only have:
  - Tree edges
  - Back edges

  - As there is no direction on the edges (can go both ways), what could be a *forward* or a *cross* edge will already have been explored in the other direction, as either a *backward* or a *tree* edge.
    - Forward (u,v) => backward (v,u)
    - Cross (u,v) => tree (v,u)

Strongly Connected Components in a Directed Graph

Strongly_Connected_Components(G)

1. `finish1`= DFS(G) //Call DFS and return the vertex finish time, `finish1`

2. Compute G^T

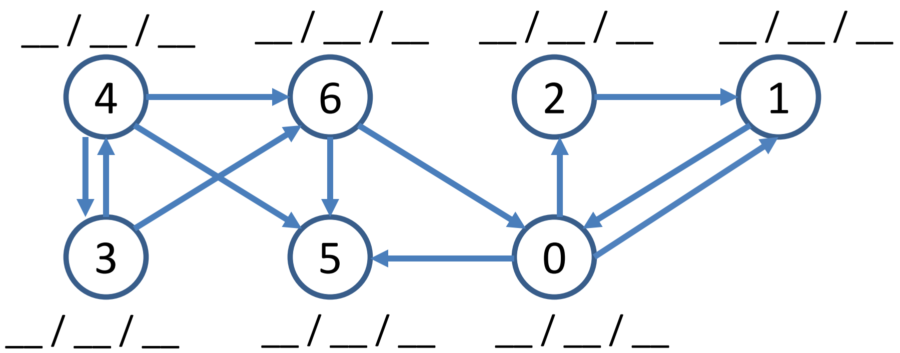
3. Call DFS(G^T), but in its main loop consider the vertices **in order of decreasing finish time,** `finish1`, (i.e. in topological order).

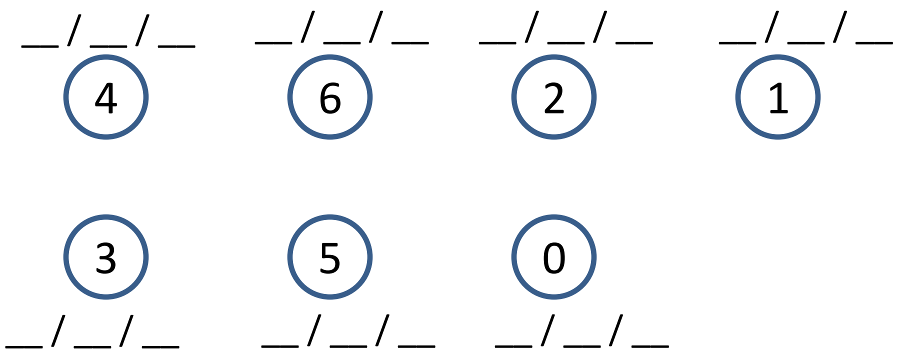
4. Output the vertices of each tree from line 3 as a separate strongly connected component.

Where: $G^T = (V, E^T)$, with $E^T = \{(v,u) : (u,v) \in E\}$

- the *transpose of G*: a graph with the same vertices as G, but with edges in reverse order.

Visited vertex	Pred	Start	Finish





Strongly Connected Components in a Directed Graph Worksheet 2

Strongly_Connected_Components(G)

1. `finish1`= DFS(G)

2. Compute G^T

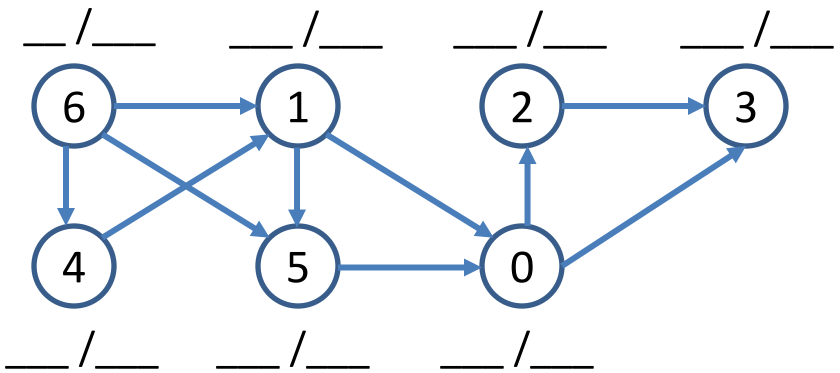
3. Call DFS(G^T), but in its main loop consider the vertices in order of decreasing finish time (`finish1`)

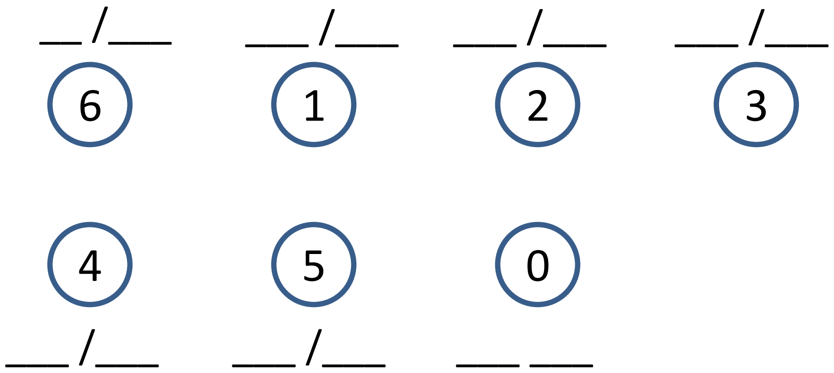
4. Output the vertices of each tree from line 3 as a separate strongly connected component.

Where: $G^T = (V, E^T)$, with $E^T = \{(v,u) : (u,v) \in E\}$

- the *transpose of G*: a graph with the same vertices as G, but with edges in reverse order.

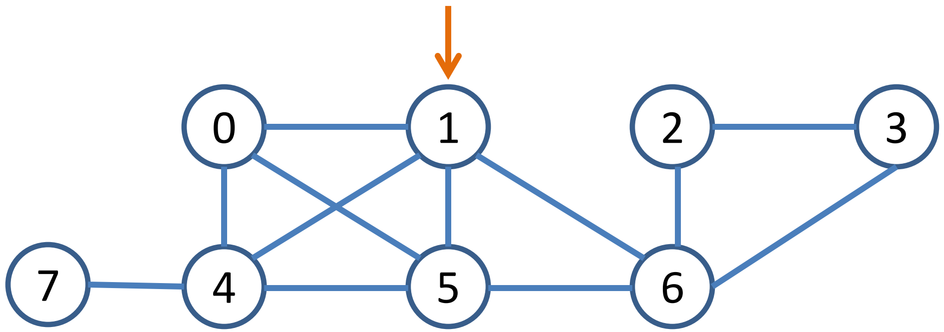
Visited vertex	Start	Finish	Pred

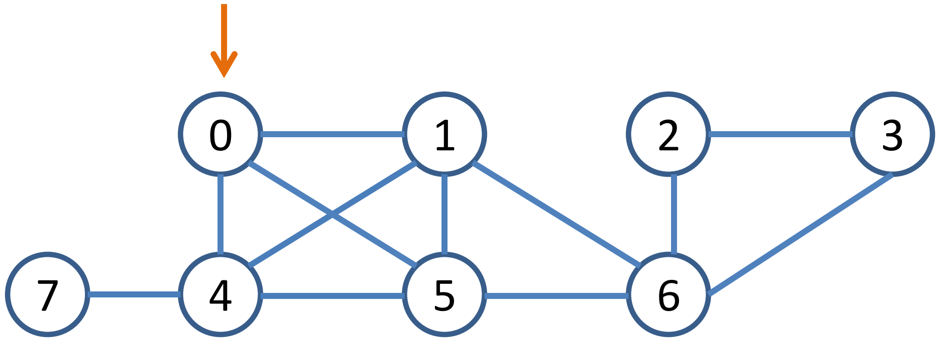




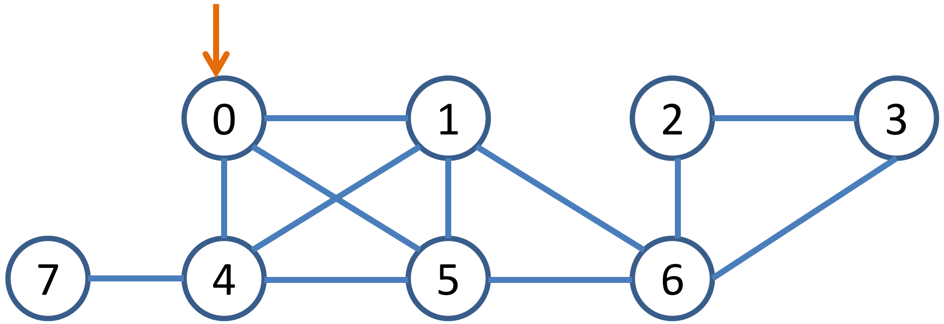
Graph Traversal - Practice

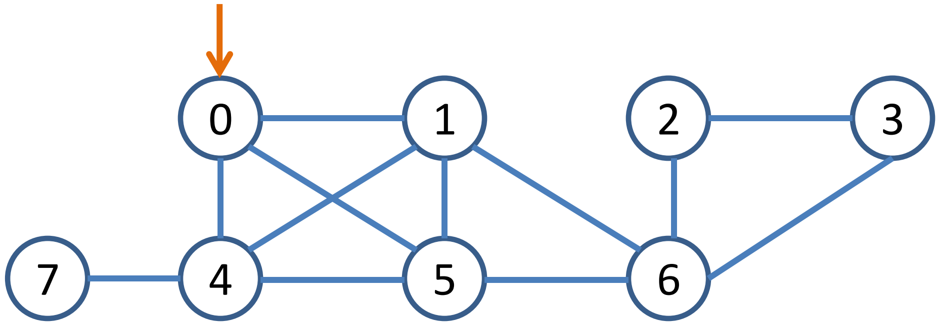
- BFS :





- DFS :

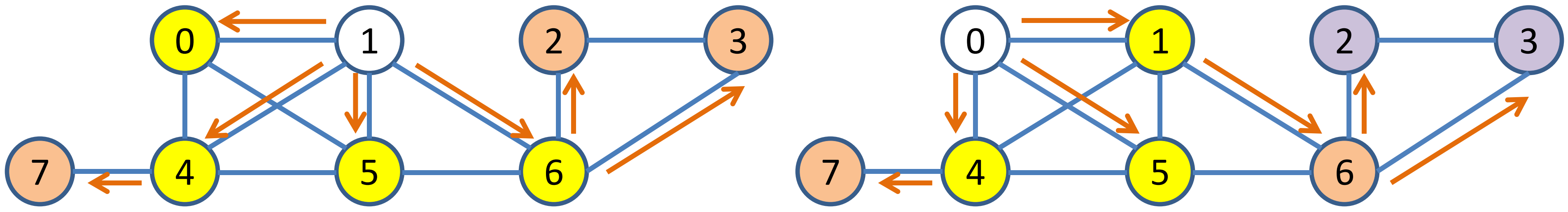




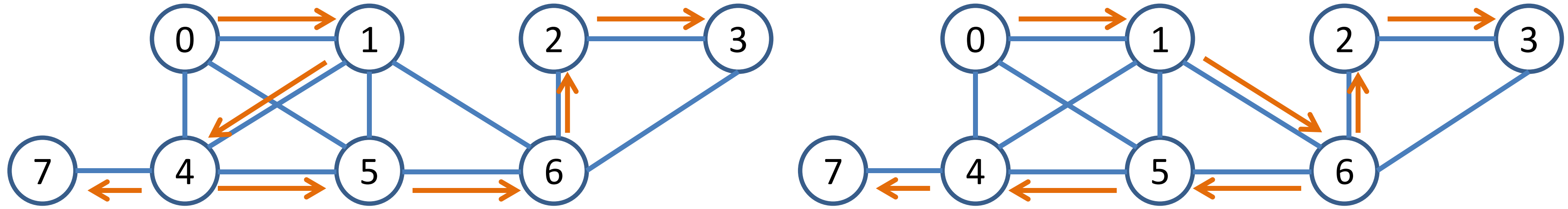
Graph Traversal

For both DFS and BFS the resulting trees depend on the order in which neighbors are visited.

- BFS traversal examples:

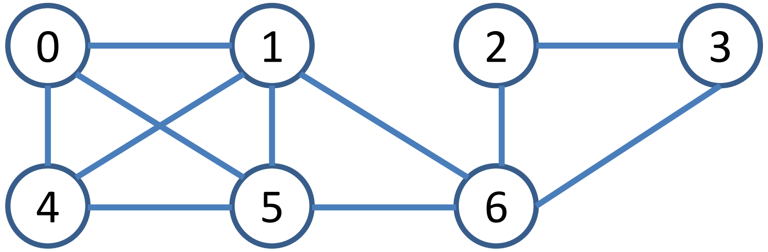


- DFS traversal examples.



DFS Practice

Try various directions for arrows.



Extra Material – Not required

DFS – Non-Recursive

- Sedgewick , Figure 5.34, page 244
- Use a stack instead of recursion
  - Visit the nodes in the same order as the recursive version.
    - Put nodes on the stack in reverse order
    - Mark node as visited when you start processing them from the stack, not when you put them on the stack
    - 2 versions based on what they put on the stack:
      - only the vertex
      - the vertex and a node reference in the adjacency list for that vertex
  - Visit the nodes in different order, but still depth-first.