Design and Analysis of Algorithms

CSE 5311

Lecture 12 Dynamic Programming

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Optimization Problems

- In which a set of choices must be made in order to arrive at an optimal (min/max) solution, subject to some constraints. (There may be several solutions to achieve *an* optimal value.)

- Two common techniques:
  - Dynamic Programming (global)
  - Greedy Algorithms (local)

Dynamic Programming (DP)

- Like divide-and-conquer, solve problem by combining the solutions to sub-problems.
- Differences between divide-and-conquer and DP:
  - Independent sub-problems, solve sub-problems independently and recursively, (so same sub(sub)problems solved repeatedly)
  - DP is applicable when the sub-problems are not independent, i.e. when sub-problems share sub-sub-problems. It solves every sub-sub-problem just once and save the results in a table to avoid duplicated computation.

Application domain of DP

- Optimization problem
  - Find a solution with optimal (maximum or minimum) value.
  - *An* optimal solution, not *the* optimal solution, since may more than one optimal solution, any one is OK.
- Typical steps
  - Characterize the structure of an optimal solution.
  - Recursively define the value of an optimal solution.
  - Compute the value of an optimal solution in a bottom-up fashion.
  - Compute an optimal solution from computed/stored information.

Elements of DP Algorithms

- **Sub-structure**: decompose problem into smaller sub-problems. Express the solution of the original problem in terms of solutions for smaller problems.
- **Table-structure**: Store the answers to the sub-problem in a table, because sub-problem solutions may be used many times.
- **Bottom-up computation**: combine solutions on smaller sub-problems to solve larger sub-problems, and eventually arrive at a solution to the complete problem.

Applicability to Optimization Problems

- **Optimal sub-structure (principle of optimality):** for the global problem to be solved optimally, each sub-problem should be solved optimally. This is often violated due to sub-problem overlaps. Often by being "less optimal" on one problem, we may make a big savings on another sub-problem.
- **Small number of sub-problems**: Many NP-hard problems can be formulated as DP problems, but these formulations are not efficient, because the number of sub-problems is exponentially large. Ideally, the number of sub-problems should be at most a polynomial number.

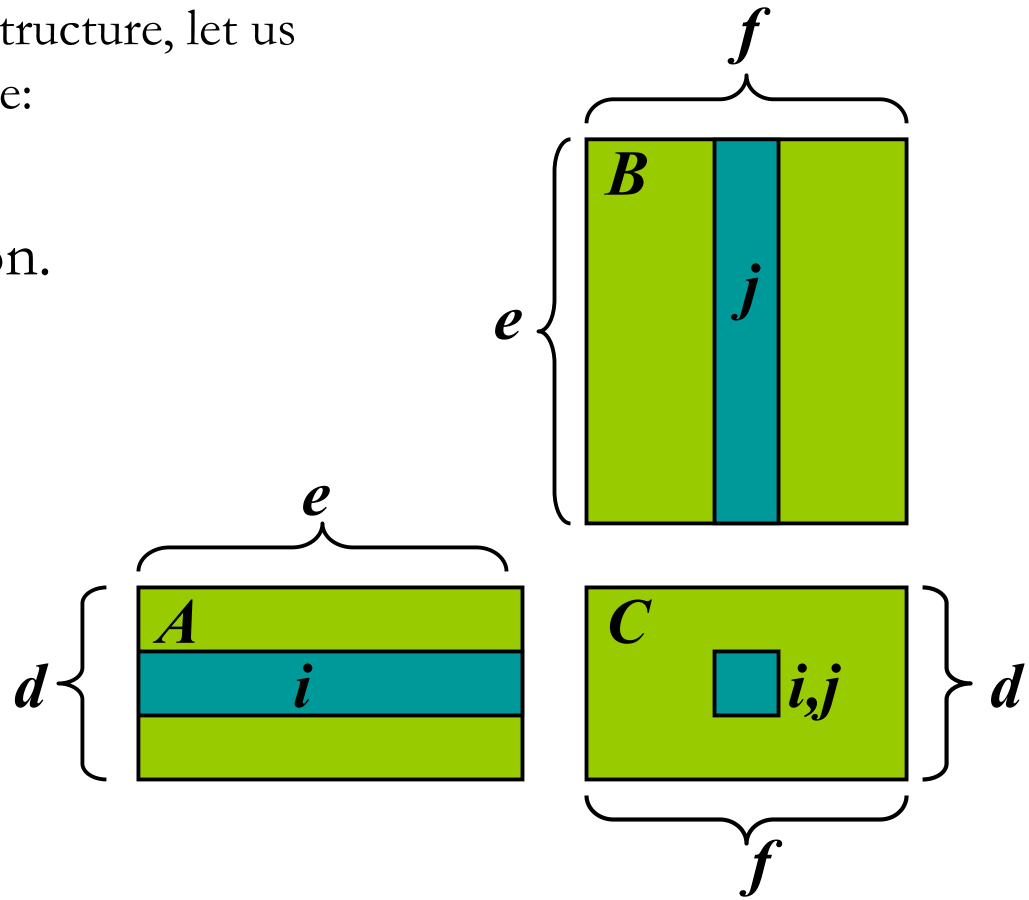
Optimized Chain Operations

- Determine the optimal sequence for performing a series of operations. (the general class of the problem is important in compiler design for code optimization & in databases for query optimization)
- For example: given a series of matrices: $A_1 \ldots A_n$, we can "parenthesize" this expression however we like, since matrix multiplication is associative (but not commutative).
- Multiply a p x q matrix A times a q x r matrix B, the result will be a p x r matrix C. (# of columns of A must be equal to # of rows of B.)

Matrix Chain-Products

- Dynamic Programming is a general algorithm design paradigm.
  - Rather than give the general structure, let us first give a motivating example:
  - **Matrix Chain-Products**
- Review: Matrix Multiplication.
  - $\boldsymbol{C} = \boldsymbol{A} * \boldsymbol{B}$
  - $\boldsymbol{A}$ is $\boldsymbol{d} \times \boldsymbol{e}$ and $\boldsymbol{B}$ is $\boldsymbol{e} \times \boldsymbol{f}$
  - $\boldsymbol{O}(\boldsymbol{d} \cdot \boldsymbol{e} \cdot \boldsymbol{f})$ time

$$C[i,j] = \sum_{k=0}^{e-1} A[i,k] * B[k,j]$$



Matrix Chain-Products

- **Matrix Chain-Product:**
  - Compute $A=A_0*A_1*\ldots*A_{n-1}$
  - A_i is $d_i \times d_{i+1}$
  - Problem: How to parenthesize?
- Example
  - B is 3×100
  - C is 100×5
  - D is 5×5
  - (B*C)*D takes 1500 + 75 = 1575 ops
  - B*(C*D) takes 1500 + 2500 = 4000 ops

Enumeration Approach

- **Matrix Chain-Product Algorithm.:**
  - Try all possible ways to parenthesize $A=A_0*A_1*\ldots*A_{n-1}$
  - Calculate number of ops for each one
  - Pick the one that is best

- Running time:
  - The number of parenthesizations is equal to the number of binary trees with n nodes
  - This is **exponential**!
  - It is called the Catalan number, and it is almost 4^n.
  - This is a terrible algorithm!

Greedy Approach

- Idea #1: repeatedly select the product that uses the fewest operations.
- Counter-example:
  - A is 101 $\times$ 11
  - B is 11 $\times$ 9
  - C is 9 $\times$ 100
  - D is 100 $\times$ 99
  - Greedy idea #1 gives A*((B*C)*D)), which takes 109989+9900+108900=228789 ops
  - (A*B)*(C*D) takes 9999+89991+89100=189090 ops
- The greedy approach is not giving us the optimal value.

"Recursive" Approach

- Define **subproblems**:
  - Find the best parenthesization of $A_i*A_{i+1}*\ldots*A_j$.
  - Let $N_{i,j}$ denote the number of operations done by this subproblem.
  - The optimal solution for the whole problem is $N_{0,n-1}$.
- **Subproblem optimality**: The optimal solution can be defined in terms of optimal subproblems
  - There has to be a final multiplication (root of the expression tree) for the optimal solution.
  - Say, the final multiplication is at index i: $(A_0*\ldots*A_i)*(A_{i+1}*\ldots*A_{n-1})$.
  - Then the optimal solution $N_{0,n-1}$ is the sum of two optimal subproblems, $N_{0,i}$ and $N_{i+1,n-1}$ plus the time for the last multiplication.

Characterizing Equation

- The global optimal has to be defined in terms of optimal subproblems, depending on where the final multiplication is at.
- Let us consider all possible places for that final multiplication:
  - Recall that A_i is a $d_i \times d_{i+1}$ dimensional matrix.
  - So, a characterizing equation for $N_{i,j}$ is the following:

$$N_{i,j} = \min_{i \le k < j} \{ N_{i,k} + N_{k+1,j} + d_i d_{k+1} d_{j+1} \}$$

- Note that subproblems are not independent–the **subproblems overlap**.

Subproblem Overlap

Algorithm ***RecursiveMatrixChain*(*S, i, j*)**:

Input: sequence ***S*** of ***n*** matrices to be multiplied

Output: number of operations in an optimal parenthesization of ***S***

if i=j

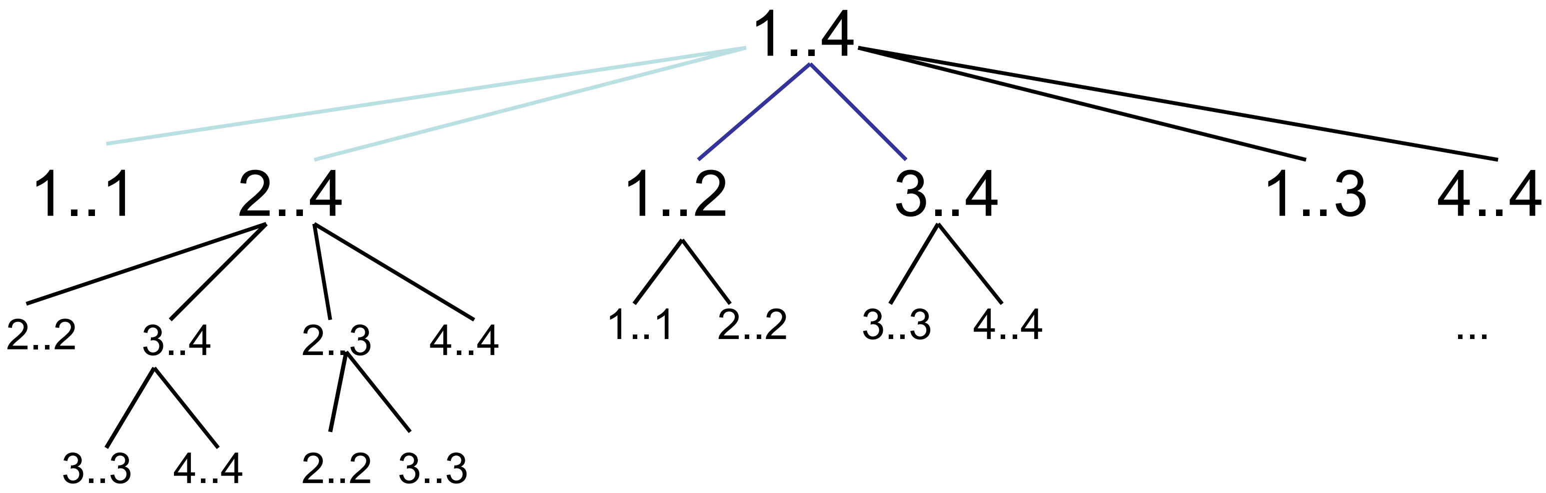
then return 0

for $k \leftarrow$ i **to** j **do**

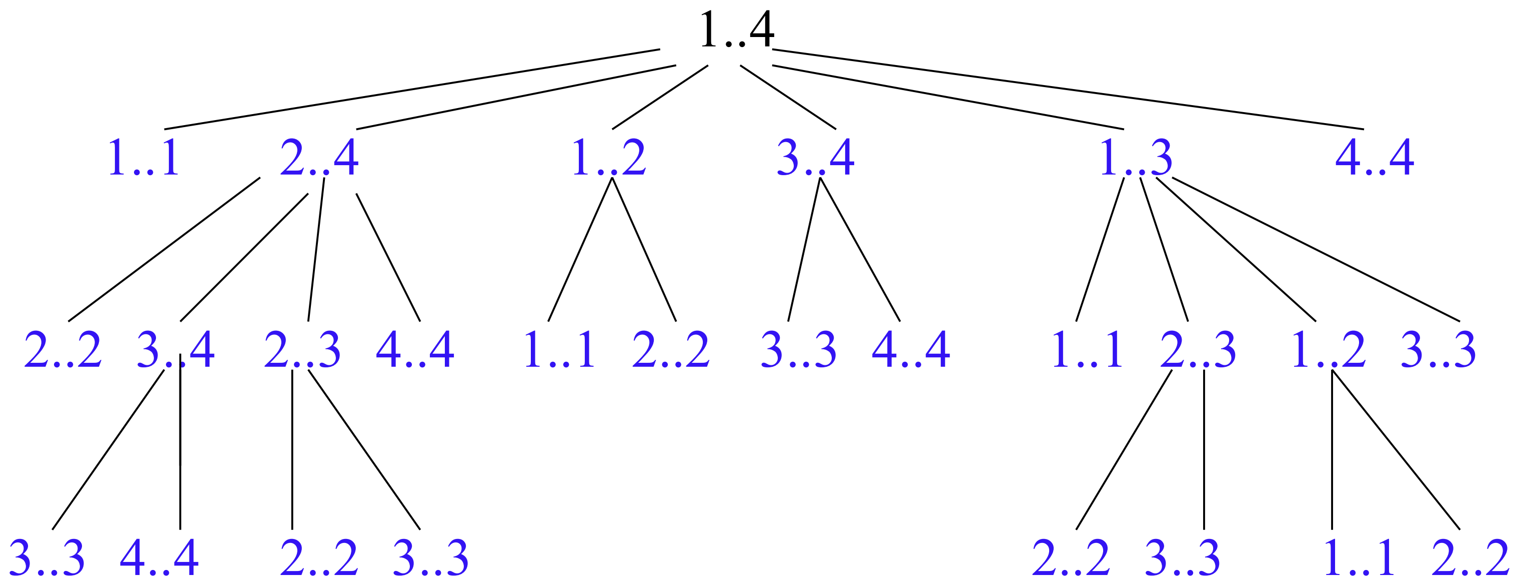
$N_{i,j} \leftarrow \min\{N_{i,j}$, ***RecursiveMatrixChain*(*S, i ,k*)**+ ***RecursiveMatrixChain*(*S, k+1,j*)**+ $d_i\, d_{k+1}\, d_{j+1}\}$

return $N_{i,j}$

Subproblem Overlap



Recursion tree for the computation of RECURSIVE-MATRIX-CHAIN(p,1,4)

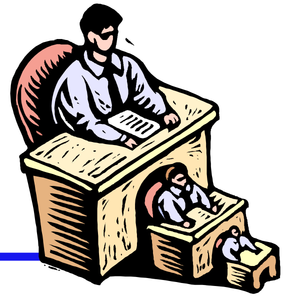


This divide-and-conquer recursive algorithm solves the overlapping problems *over and over*.

In contrast, DP solves the same (overlapping) subproblems only once (at the first time), then store the result in a table, when the same subproblem is encountered later, just look up the table to get the result.

The computations in green color are replaced by table look up in MEMOIZED-MATRIX-CHAIN(p,1,4)
The divide-and-conquer is better for the problem which generates brand-new problems at each step of recursion.

Dynamic Programming Algorithm



- Since subproblems overlap, we don't use recursion.
- Instead, we construct optimal subproblems "bottom-up."
- $N_{i,i}$'s are easy, so start with them
- Then do problems of "length" 2,3,… subproblems, and so on.
- Running time: $O(n^3)$

Algorithm ***matrixChain***(***S***):

Input: sequence S of n matrices to be multiplied

Output: number of operations in an optimal parenthesization of S

for $i \leftarrow 1$ **to** $n - 1$ **do**

    $N_{i,i} \leftarrow 0$

for $b \leftarrow 1$ **to** $n - 1$ **do**

    { $b = j - i$ is the length of the problem }

    for $i \leftarrow 0$ **to** $n - b - 1$ **do**

        $j \leftarrow i + b$

        $N_{i,j} \leftarrow +\infty$

        for $k \leftarrow i$ **to** $j - 1$ **do**

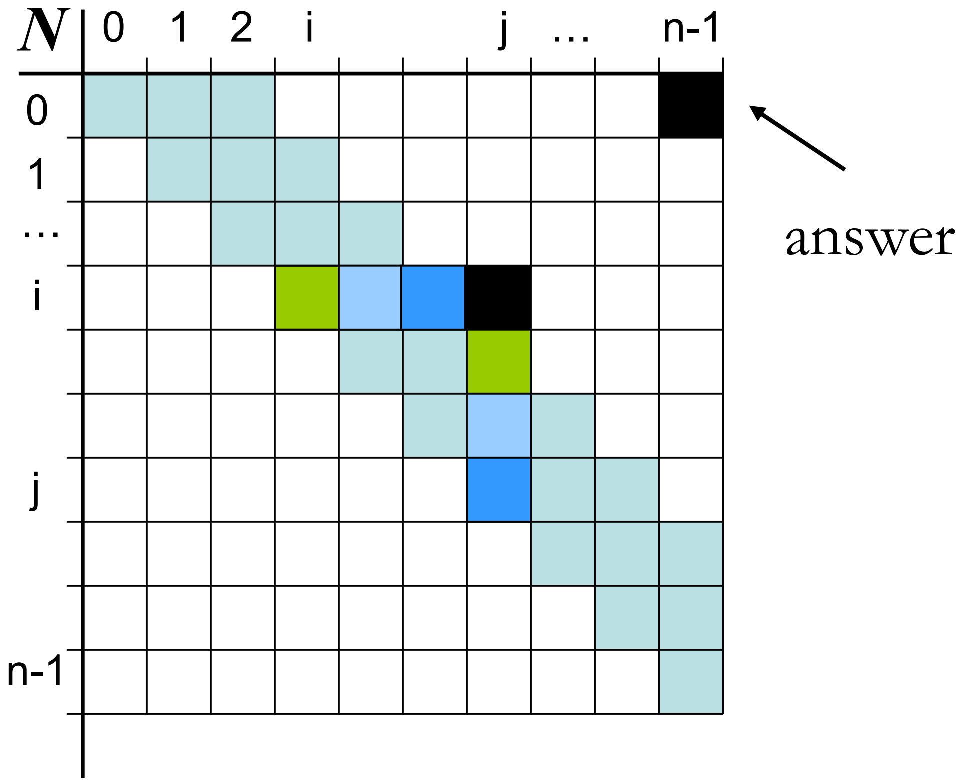
            $N_{i,j} \leftarrow \min\{N_{i,j}, N_{i,k} + N_{k+1,j} + d_i d_{k+1} d_{j+1}\}$

return $N_{0,n-1}$

Dynamic Programming Algorithm Visualization

- The bottom-up construction fills in the N array by diagonals
- $N_{i,j}$ gets values from previous entries in i-th row and j-th column
- Filling in each entry in the N table takes O(n) time.
- Total run time: $O(n^3)$
- Getting actual parenthesization can be done by remembering "k" for each N entry

$$N_{i,j} = \min_{i \le k < j}\{N_{i,k} + N_{k+1,j} + d_i d_{k+1} d_{j+1}\}$$



Dynamic Programming Algorithm Visualization

- A_0: 30 X 35; A_1: 35 X15; A_2: 15X5;
  A_3: 5X10; A_4: 10X20; A_5: 20 X 25

0	1	2	3	4	5	
0	15,750	7,875	9,375	11,875	15,125	0
	0	2,625	4,375	7,125	10,500	1
		0	750	2,500	5,375	2
			0	1,000	3,500	3
				0	5,000	4
					0	5

$$N_{i,j} = \min_{i \le k < j}\{N_{i,k} + N_{k+1,j} + d_i d_{k+1} d_{j+1}\}$$

$$\begin{aligned}
&N_{1,4} = \min\{\\
&N_{1,1} + N_{2,4} + d_1 d_2 d_5 = 0 + 2500 + 35*15*20 = 13000,\\
&N_{1,2} + N_{3,4} + d_1 d_3 d_5 = 2625 + 1000 + 35*5*20 = 7125,\\
&N_{1,3} + N_{4,4} + d_1 d_4 d_5 = 4375 + 0 + 35*10*20 = 11375\\
&\}\\
&= 7125
\end{aligned}$$

Dynamic Programming Algorithm Visualization

0	1	2	3	4	5	
0	15,750	7,875	9,375	11,875	15,125	0
	0	2,625	4,375	7,125	10,500	1
		0	750	2,500	5,375	2
			0	1,000	3,500	3
				0	5,000	4
					0	5

0	1	2	3	4	5	
	0	0	2	2	2	0
		1	2	2	2	1
			2	2	2	2
				3	4	3
					4	4
						5

$(A_0*(A_1*A_2))*((A_3*A_4)*A_5)$

Assembly-Line Scheduling

- Two parallel assembly lines in a factory, lines 1 and 2
- Each line has n stations $S_{i,1} \ldots S_{i,n}$
- For each j, $S_{1,j}$ does the same thing as $S_{2,j}$, but it may take a different amount of assembly time $a_{i,j}$
- Transferring away from line i after stage j costs $t_{i,j}$
- Also entry time e_i and exit time x_i at beginning and end

Assembly Line Scheduling (ALS)

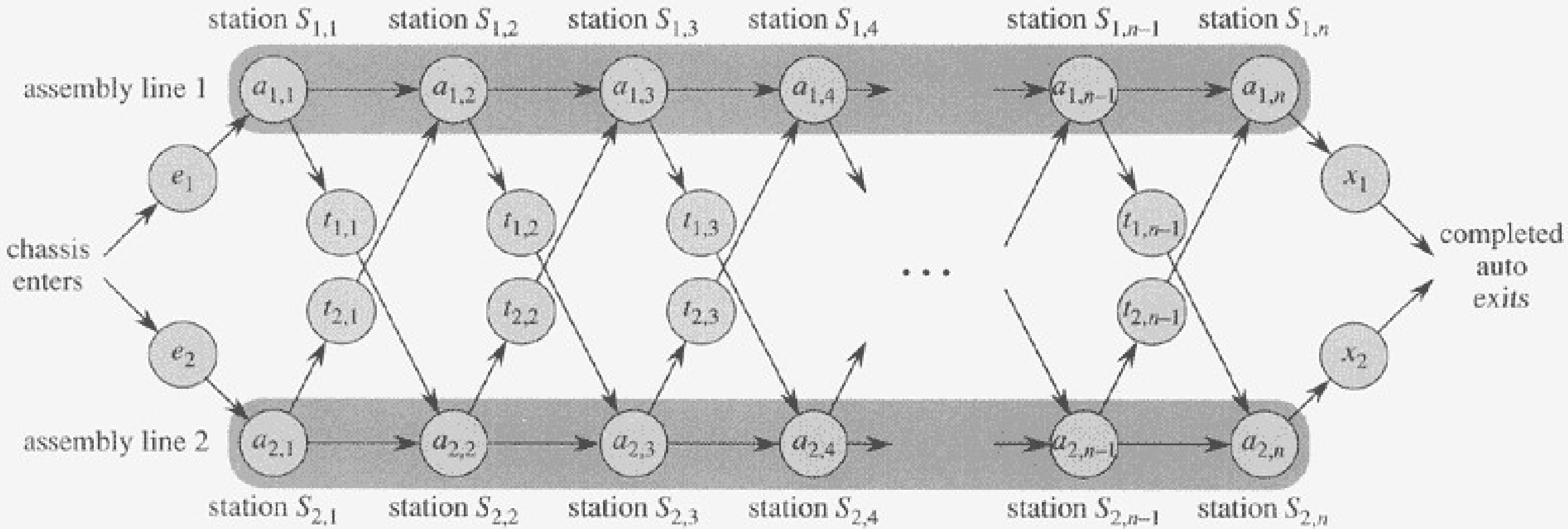
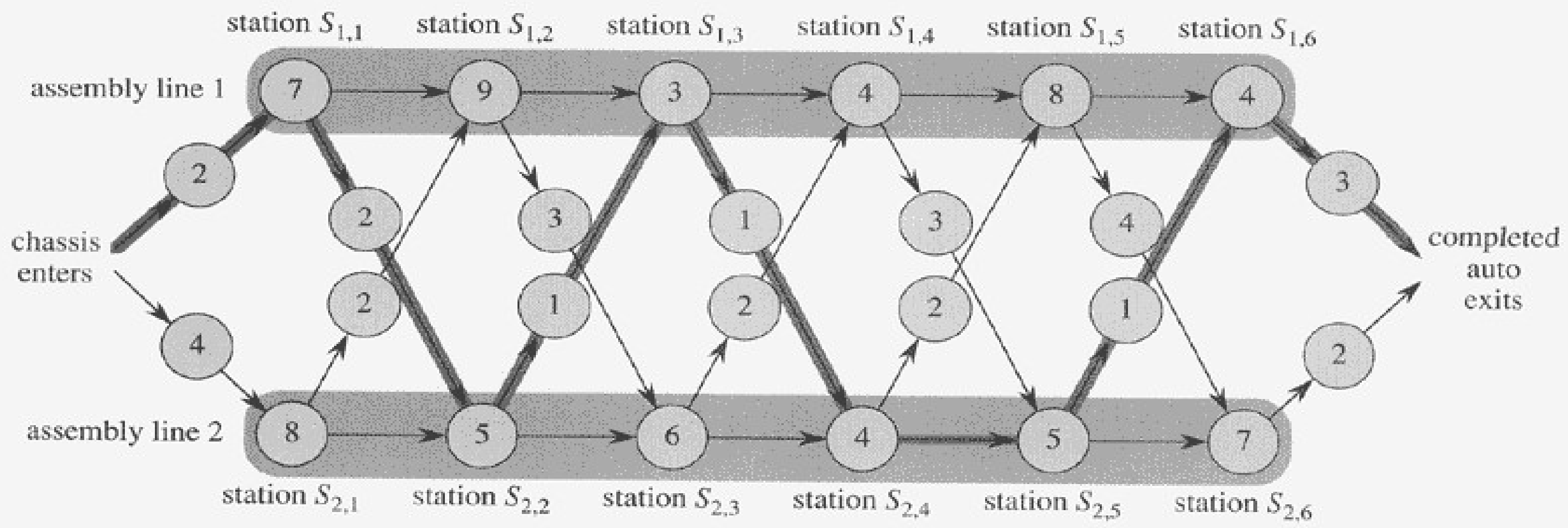


Figure 15.1 A manufacturing problem to find the fastest way through a factory. There are two assembly lines, each with n stations; the jth station on line i is denoted $S_{i,j}$ and the assembly time at that station is $a_{i,j}$. An automobile chassis enters the factory, and goes onto line i (where $i = 1$ or 2), taking e_i time. After going through the jth station on a line, the chassis goes on to the $(j+1)$st station on either line. There is no transfer cost if it stays on the same line, but it takes time $t_{i,j}$ to transfer to the other line after station $S_{i,j}$. After exiting the nth station on a line, it takes x_i time for the completed auto to exit the factory. The problem is to determine which stations to choose from line 1 and which to choose from line 2 in order to minimize the total time through the factory for one auto.

Concrete Instance of ALS



(a)

j	1	2	3	4	5	6
$f_1[j]$	9	18	20	24	32	35
$f_2[j]$	12	16	22	25	30	37

$f^* = 38$

j	2	3	4	5	6
$l_1[j]$	1	2	1	1	2
$l_2[j]$	1	2	1	2	2

$l^* = 1$

(b)

Figure 15.2 **(a)** An instance of the assembly-line problem with costs e_i, $a_{i,j}$, $t_{i,j}$, and x_i indicated. The heavily shaded path indicates the fastest way through the factory. **(b)** The values of $f_i[j]$, f^*, $l_i[j]$, and l^* for the instance in part (a).

Brute Force Solution

– List all possible sequences,
– For each sequence of n stations, compute the passing time. (the computation takes $\Theta(n)$ time.)
– Record the sequence with smaller passing time.
– However, there are total 2^n possible sequences.

ALS --DP steps: Step 1

- Step 1: find the structure of the fastest way through factory
  - Consider the fastest way from starting point through station $S_{1,j}$ (same for $S_{2,j}$)
    - ➢ j=1, only one possibility
    - ➢ j=2,3,…,n, two possibilities: from $S_{1,j-1}$ or $S_{2,j-1}$
      - from $S_{1,j-1}$, additional time $a_{1,j}$
      - from $S_{2,j-1}$, additional time $t_{2,j-1} + a_{1,j}$
    - ➢ suppose the fastest way through $S_{1,j}$ is through $S_{1,j-1}$, then the chassis must have taken a fastest way from starting point through $S_{1,j-1}$. Why???
    - ➢ Similarly for $S_{2,j-1}$.

DP step 1: Find Optimal Structure

- An optimal solution to a problem contains within it an optimal solution to subproblems.
- the fastest way through station $S_{i,j}$ contains within it the fastest way through station $S_{1,j-1}$ or $S_{2,j-1}$.
- Thus can construct an optimal solution to a problem from the optimal solutions to subproblems.

ALS --DP steps: Step 2

- Step 2: A recursive solution
- Let $f_i[j]$ (i=1,2 and j=1,2,…, n) denote the fastest possible time to get a chassis from starting point through $S_{i,j}$.
- Let f^* denote the fastest time for a chassis all the way through the factory. Then
- $f^* = \min(f_1[n] + x_1, f_2[n] + x_2)$
- $f_1[1]=e_1+a_{1,1}$, fastest time to get through $S_{1,1}$
- $f_1[j]=\min(f_1[j\text{-}1]+a_{1,j}, f_2[j\text{-}1]+ t_{2,j\text{-}1}+ a_{1,j})$
- Similarly to $f_2[j]$.

ALS --DP steps: Step 2

- Recursive solution:
  - $f^* = \min(f_1[n] + x_1, f_2[n] + x_2)$
  - $f_1[j] = \begin{cases} e_1 + a_{1,1} & \text{if } j=1 \\ \min(f_1[j-1] + a_{1,j}, f_2[j-1] + t_{2,j-1} + a_{1,j}) & \text{if } j>1 \end{cases}$
  - $f_2[j] = \begin{cases} e_2 + a_{2,1} & \text{if } j=1 \\ \min(f_2[j-1] + a_{2,j}, f_1[j-1] + t_{1,j-1} + a_{2,j}) & \text{if } j>1 \end{cases}$
- $f_i[j]$ (i=1,2; j=1,2,…,n) records optimal values to the subproblems.
- To keep track of the fastest way, introduce $l_i[j]$ to record the line number (1 or 2), whose station j-1 is used in a fastest way through $S_{i,j}$.
- Introduce l^* to be the line whose station n is used in a fastest way through the factory.

ALS --DP steps: Step 3

- Step 3: Computing the fastest time
  - One option: a recursive algorithm.
    - ➢Let $r_i(j)$ be the number of references made to $f_i[j]$
      - $r_1(n) = r_2(n) = 1$
      - $r_1(j) = r_2(j) = r_1(j+1) + r_2(j+1)$
      - $r_i(j) = 2^{n-j}$.
      - So $f_1[1]$ is referred to 2^{n-1} times.
      - Total references to all $f_i[j]$ is $\Theta(2^n)$.
    - ➢Thus, the running time is exponential.
  - Non-recursive algorithm.

ALS FAST-WAY Algorithm

```
FASTEST-WAY(a, t, e, x, n)
f_1[1] ← e_1 + a_{1,1}
f_2[1] ← e_2 + a_{2,1}
for j ← 2 to n
    do if f_1[j − 1] + a_{1,j} ≤ f_2[j − 1] + t_{2,j−1} + a_{1,j}
          then f_1[j] ← f_1[j − 1] + a_{1,j}
               l_1[j] ← 1
          else f_1[j] ← f_2[j − 1] + t_{2,j−1} + a_{1,j}
               l_1[j] ← 2
       if f_2[j − 1] + a_{2,j} ≤ f_1[j − 1] + t_{1,j−1} + a_{2,j}
          then f_2[j] ← f_2[j − 1] + a_{2,j}
               l_2[j] ← 2
          else f_2[j] ← f_1[j − 1] + t_{1,j−1} + a_{2,j}
               l_2[j] ← 1
if f_1[n] + x_1 ≤ f_2[n] + x_2
   then f* = f_1[n] + x_1
        l* = 1
   else f* = f_2[n] + x_2
        l* = 2
```

Running time: *O*(*n*).

ALS --DP steps: Step 4

- **Step 4:** Construct the fastest way through the factory

```
PRINT-STATIONS(l, n)
i ← l*
print "line " i ", station " n
for j ← n downto 2
        do i ← l_i[j]
            print "line " i ", station " j − 1
```

Optimal Substructure Varies in Two Ways

- How many subproblems
  - In assembly-line schedule, one subproblem
  - In matrix-chain multiplication: two subproblems
- How many choices
  - In assembly-line schedule, two choices
  - In matrix-chain multiplication: j-i choices
- DP solve the problem in bottom-up manner.

Running Time for DP Programs

- #overall subproblems × #choices.
  - In assembly-line scheduling, $O(n) \times O(1) = O(n)$.
  - In matrix-chain multiplication, $O(n^2) \times O(n) = O(n^3)$
- The cost =costs of solving subproblems + cost of making choice.
  - In assembly-line scheduling, choice cost is
    - $a_{i,j}$ if stay in the same line, $t_{i',j-1}+a_{i,j}$ $(i' \neq i)$ otherwise.
  - In matrix-chain multiplication, choice cost is $p_{i-1}p_k p_j$.