Design and Analysis of Algorithms

CSE 5311

Lecture 14 Dynamic Programming

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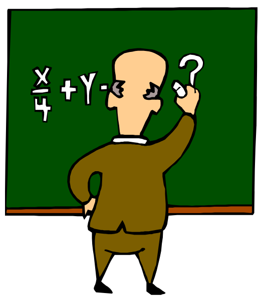
The General Dynamic Programming Technique

- Applies to a problem that at first seems to require a lot of time (possibly exponential), provided we have:
  - **Subproblem optimality:** the global optimum value can be defined in terms of optimal subproblems
  - **Subproblem overlap:** the subproblems are not independent, but instead they overlap (hence, should be constructed bottom-up).



Longest Common Subsequence

- ***Problem:*** Given 2 sequences, $X = \langle x_1,\ldots,x_m \rangle$ and $Y = \langle y_1,\ldots,y_n \rangle$, find a common subsequence whose length is maximum.

Subsequence **need not be consecutive**, but **must be in order.**

Other Sequence Questions

- ***Edit distance:*** Given 2 sequences, $X = \langle x_1,\ldots,x_m \rangle$ and $Y = \langle y_1,\ldots,y_n \rangle$, what is the minimum number of deletions, insertions, and changes that you must do to change one to another?
- ***Protein sequence alignment:*** Given a score matrix on amino acid pairs, $s(a,b)$ for $a,b \in \{\Lambda\} \cup A$, and 2 amino acid sequences, $X = \langle x_1,\ldots,x_m \rangle \in A^m$ and $Y = \langle y_1,\ldots,y_n \rangle \in A^n$, find the alignment with lowest score…

More Problems

Optimal BST: Given sequence $K = k_1 < k_2 < \cdots < k_n$ of n sorted keys, with a search probability p_i for each key k_i, build a binary search tree (BST) with minimum expected search cost.

Minimum convex decomposition of a polygon,

Hydrogen placement in protein structures, …

Dynamic Programming

- Dynamic Programming is an algorithm design technique for optimization problems: often minimizing or maximizing.
- Like divide and conquer, DP solves problems by combining solutions to subproblems.
- Unlike divide and conquer, subproblems are not independent.
  - Subproblems may share subsubproblems,
  - However, solution to one subproblem may not affect the solutions to other subproblems of the same problem. (More on this later.)
- DP reduces computation by
  - Solving subproblems in a bottom-up fashion.
  - Storing solution to a subproblem the first time it is solved.
  - Looking up the solution when subproblem is encountered again.
- Key: determine structure of optimal solutions

Recalling: Steps in Dynamic Programming

1. Characterize structure of an optimal solution.
2. Define value of optimal solution recursively.
3. Compute optimal solution values either top-down with caching or bottom-up in a table.
4. Construct an optimal solution from computed values.

Naïve Algorithm

- For every subsequence of $X = \langle x_1,\ldots,x_m \rangle$, check whether it's a subsequence of $Y = \langle y_1,\ldots,y_n \rangle$.
- Time: $\Theta(n2^m)$.
  - 2^m subsequences of X to check.
  - Each subsequence takes $\Theta(n)$ time to check: scan Y for first letter, for second, and so on.

Optimal Substructure

Theorem

Let $Z = \langle z_1, \ldots, z_k \rangle$ be any LCS of X and Y.

1. If $x_m = y_n$, then $z_k = x_m = y_n$ and Z_{k-1} is an LCS of X_{m-1} and Y_{n-1}.
2. If $x_m \neq y_n$, then either $z_k \neq x_m$ and Z is an LCS of X_{m-1} and Y.
3. or $z_k \neq y_n$ and Z is an LCS of X and Y_{n-1}.

Notation:

prefix $X_i = \langle x_1, \ldots, x_i \rangle$ is the first i letters of X.

This says what any longest common subsequence must look like; do you believe it?

Optimal Substructure

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3. or $z_k \neq y_n$ and Z is an LCS of X and Y_{n-1}.

Proof: (case 1: $x_m = y_n$)

Any sequence Z' that does not end in $x_m = y_n$ can be made longer by adding $x_m = y_n$ to the end. Therefore,

(1) longest common subsequence (LCS) Z must end in $x_m = y_n$.

(2) Z_{k-1} is a common subsequence of X_{m-1} and Y_{n-1}, and

(3) there is no longer CS of X_{m-1} and Y_{n-1}, or Z would not be an LCS.

Optimal Substructure

Theorem

Let $Z = \langle z_1, \ldots, z_k \rangle$ be any LCS of X and Y.

1. If $x_m = y_n$, then $z_k = x_m = y_n$ and Z_{k-1} is an LCS of X_{m-1} and Y_{n-1}.
2. If $x_m \neq y_n$, then either $z_k \neq x_m$ and Z is an LCS of X_{m-1} and Y.
3. or $z_k \neq y_n$ and Z is an LCS of X and Y_{n-1}.

Proof: (case 2: $x_m \neq y_n$, and $z_k \neq x_m$)

Since Z does not end in x_m,

(1) Z is a common subsequence of X_{m-1} and Y, and

(2) there is no longer CS of X_{m-1} and Y, or Z would not be an LCS.

Recursive Solution

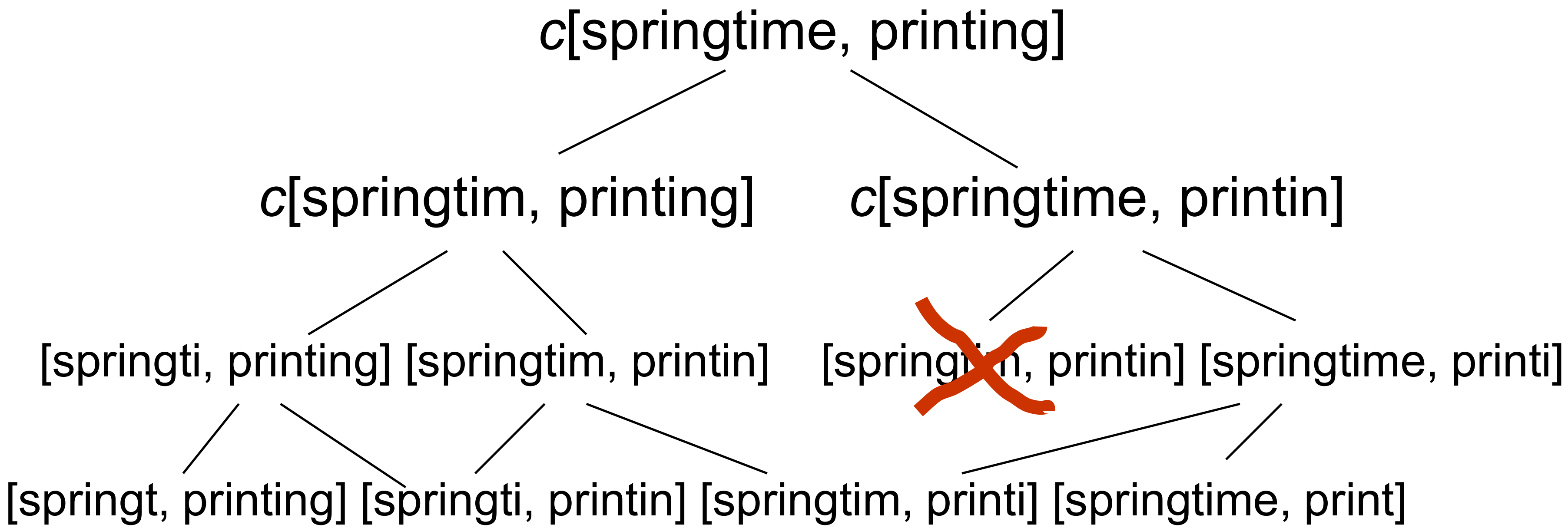
- Define $c[i, j]$ = length of LCS of X_i and Y_j .
- We want $c[m,n]$.

$$c[i,j]=\begin{cases}0 & \text{if } i=0 \text{ or } j=0,\\ c[i-1,j-1]+1 & \text{if } i,j>0 \text{ and } x_i=y_j,\\ \max(c[i-1,j],c[i,j-1]) & \text{if } i,j>0 \text{ and } x_i\neq y_j.\end{cases}$$

This gives a recursive algorithm and solves the problem.
But does it solve it well?

Recursive Solution

$$c[\alpha, \beta] = \begin{cases} 0 & \text{if } \alpha \text{ empty or } \beta \text{ empty,} \\ c[prefix\,\alpha, prefix\,\beta] + 1 & \text{if end}(\alpha) = \text{end}(\beta), \\ \max(c[prefix\,\alpha, \beta], c[\alpha, prefix\,\beta]) & \text{if end}(\alpha) \neq \text{end}(\beta). \end{cases}$$



Recursive Solution

$$c[\alpha,\beta]=\begin{cases}0 & \text{if } \alpha \text{ empty or } \beta \text{ empty },\\ c[\mathit{prefix}\ \alpha, \mathit{prefix}\ \beta]+1 & \text{if end}(\alpha)=\text{end}(\beta),\\ \max(c[\mathit{prefix}\ \alpha,\beta], c[\alpha, \mathit{prefix}\ \beta]) & \text{if end}(\alpha)\neq\text{end}(\beta).\end{cases}$$

- Keep track of *c*[*a*,*b*] in a table of *nm* entries:
  - top/down
  - bottom/up

		p	r	i	n	t	i	n	g
S									
P									
r									
i									
n									
g									
t									
i									
m									
e									

Computing the length of an LCS

```
LCS-LENGTH (X, Y)
m ← length[X]
n ← length[Y]
for i ← 1 to m
      do c[i, 0] ← 0
for j ← 0 to n
      do c[0, j ] ← 0
for i ← 1 to m
       do for j ← 1 to n
            do if x_i = y_j
                      then c[i, j ] ← c[i–1, j–1] + 1
                              b[i, j ] ← "↖"
                      else if c[i–1, j ] ≥ c[i, j–1]
                              then c[i, j ] ← c[i– 1, j ]
                                      b[i, j ] ← "↑"
                              else c[i, j ] ← c[i, j–1]
                                      b[i, j ] ← "←"
return c and b
```

$b[i, j]$ points to table entry whose subproblem we used in solving LCS of X_i and Y_j.

$c[m,n]$ contains the length of an LCS of X and Y.

Time: $O(mn)$

Constructing an LCS

<u>PRINT-LCS (b, X, i, j)</u>

1. **if** $i = 0$ or $j = 0$
2. **then return**
3. **if** $b[i, j] =$ "↖"
4. **then** PRINT-LCS($b, X, i–1, j–1$)
5. print x_i
6. **elseif** $b[i, j] =$ "↑"
7. **then** PRINT-LCS($b, X, i–1, j$)
8. **else** PRINT-LCS($b, X, i, j–1$)

- Initial call is PRINT-LCS (b, X, m, n).
- When $b[i, j] =$ ↖ , we have extended LCS by one character. So LCS = entries with ↖ in them.
- Time: $O(m+n)$

LCS Example

We'll see how LCS algorithm works on the following example:

- X = ABCB
- Y = BDCAB

What is the Longest Common Subsequence of X and Y?

LCS(X, Y) = BCB

X = A **B** **C** **B**

Y = **B** D **C** A **B**

LCS Example (0)

ABCB
BDCAB

	j	0	1	2	3	4	5
i		Yj	**B**	**D**	**C**	**A**	**B**
0	Xi						
1	**A**						
2	**B**						
3	**C**						
4	**B**						

X = ABCB;   m = |X| = 4
Y = BDCAB; n = |Y| = 5
Allocate array c[5,4]

LCS Example (1)

ABCB
BDCAB

i		j	0	1	2	3	4	5
			Yj	B	D	C	A	B
0	Xi		0	0	0	0	0	0
1	A		0					
2	B		0					
3	C		0					
4	B		0					

for i = 1 to m c[i,0] = 0
for j = 1 to n c[0,j] = 0

LCS Example (2)

ABCB
BDCAB

	j	0	1	2	3	4	5
i		Yj	B	D	C	A	B
0	Xi	0	0	0	0	0	0
1	A	0	0				
2	B	0					
3	C	0					
4	B	0					

if (X_i == Y_j)
    c[i,j] = c[i-1,j-1] + 1
else c[i,j] = max(c[i-1,j], c[i,j-1])

LCS Example (3)

ABCB
BDCAB

j		0	1	2	3	4	5
i		Yj	B	D	C	A	B
0	Xi	0	0	0	0	0	0
1	A	0	0	0	0		
2	B	0					
3	C	0					
4	B	0					

if ($X_i == Y_j$)
    $c[i,j] = c[i-1,j-1] + 1$
else $c[i,j] = \max(c[i-1,j], c[i,j-1])$

LCS Example (4)

ABCB
BDCAB

	j	0	1	2	3	4	5
i		Yj	B	D	C	A	B
0	Xi	0	0	0	0	0	0
1	A	0	0	0	0	1	
2	B	0					
3	C	0					
4	B	0					

if (X_i == Y_j)
    c[i,j] = c[i-1,j-1] + 1
else c[i,j] = max(c[i-1,j], c[i,j-1])

LCS Example (5)

ABCB
BDCAB

i		j: 0	1	2	3	4	5
		Yj	B	D	C	A	B
0	Xi	0	0	0	0	0	0
1	A	0	0	0	0	1 →	1
2	B	0					
3	C	0					
4	B	0					

if (X_i == Y_j)
c[i,j] = c[i-1,j-1] + 1
else c[i,j] = max(c[i-1,j], c[i,j-1])

LCS Example (6)

ABCB
BDCAB

	j	0	1	2	3	4	5
i		Yj	B	D	C	A	B
0	Xi	0	0	0	0	0	0
1	A	0	0	0	0	1	1
2	B	0	1				
3	C	0					
4	B	0					

if ($X_i == Y_j$)
    c[i,j] = c[i-1,j-1] + 1
else c[i,j] = max(c[i-1,j], c[i,j-1])

LCS Example (7)

ABCB
BDCAB

	j	0	1	2	3	4	5
i		Yj	B	D	C	A	B
0	Xi	0	0	0	0	0	0
1	A	0	0	0	0	1	1
2	B	0	1	1	1	1	
3	C	0					
4	B	0					

if (X_i == Y_j)
c[i,j] = c[i-1,j-1] + 1
else c[i,j] = max(c[i-1,j], c[i,j-1])

LCS Example (8)

ABCB
BDCAB

i		j	0	1	2	3	4	5
		Yj	B	D	C	A	B
0	Xi	0	0	0	0	0	0
1	A	0	0	0	0	1	1
2	B	0	1	1	1	1	2
3	C	0					
4	B	0					

if ($X_i == Y_j$)
c[i,j] = c[i-1,j-1] + 1
else c[i,j] = max(c[i-1,j], c[i,j-1])

LCS Example (10)

ABCB
BDCAB

j		0	1	2	3	4	5
i		Yj	B	D	C	A	B
0	Xi	0	0	0	0	0	0
1	A	0	0	0	0	1	1
2	B	0	1	1	1	1	2
3	C	0	1	1			
4	B	0					

if ($X_i == Y_j$)
$c[i,j] = c[i-1,j-1] + 1$
else $c[i,j] = \max(c[i-1,j], c[i,j-1])$

LCS Example (11)

ABCB
BDCAB

j		0	1	2	3	4	5
i		Yj	B	D	C	A	B
0	Xi	0	0	0	0	0	0
1	A	0	0	0	0	1	1
2	B	0	1	1	1	1	2
3	C	0	1	1	2		
4	B	0					

if (X_i == Y_j)
c[i,j] = c[i-1,j-1] + 1
else c[i,j] = max(c[i-1,j], c[i,j-1])

LCS Example (12)

ABCB
BDCAB

j		0	1	2	3	4	5
i		Yj	B	D	C	A	B
0	Xi	0	0	0	0	0	0
1	A	0	0	0	0	1	1
2	B	0	1	1	1	1	2
3	C	0	1	1	2	2	2
4	B	0					

if ($X_i == Y_j$)
c[i,j] = c[i-1,j-1] + 1
else c[i,j] = max(c[i-1,j], c[i,j-1])

LCS Example (13)

ABCB
BDCAB

	j	0	1	2	3	4	5
i		Yj	B	D	C	A	B
0	Xi	0	0	0	0	0	0
1	A	0	0	0	0	1	1
2	B	0	1	1	1	1	2
3	C	0	1	1	2	2	2
4	B	0	1				

if (X_i == Y_j)
c[i,j] = c[i-1,j-1] + 1
else c[i,j] = max(c[i-1,j], c[i,j-1])

LCS Example (14)

ABCB
BDCAB

j		0	1	2	3	4	5
i		Yj	B	D	C	A	B
0	Xi	0	0	0	0	0	0
1	A	0	0	0	0	1	1
2	B	0	1	1	1	1	2
3	C	0	1	1	2	2	2
4	B	0	1	1	2	2	

if ($X_i == Y_j$)
  $c[i,j] = c[i-1,j-1] + 1$
else $c[i,j] = \max(c[i-1,j], c[i,j-1])$

LCS Example (15)

ABCB
BDCAB

	j	0	1	2	3	4	5
i		Yj	B	D	C	A	B
0	Xi	0	0	0	0	0	0
1	A	0	0	0	0	1	1
2	B	0	1	1	1	1	2
3	C	0	1	1	2	2	2
4	B	0	1	1	2	2	3

if (X_i == Y_j)
c[i,j] = c[i-1,j-1] + 1
else c[i,j] = max(c[i-1,j], c[i,j-1])

LCS Algorithm Running Time

- LCS algorithm calculates the values of each entry of the array c[m,n]
- So what is the running time?

O(m*n)

since each c[i,j] is calculated in constant time, and there are m*n elements in the array

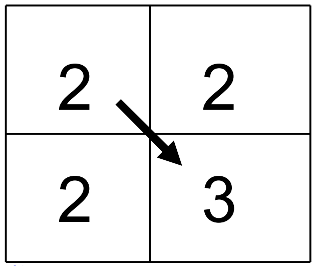
How to find actual LCS

- So far, we have just found the *length* of LCS, but not LCS itself.
- We want to modify this algorithm to make it output Longest Common Subsequence of X and Y

Each *c[i,j]* depends on *c[i-1,j]* and *c[i,j-1]*

or *c[i-1, j-1]*

For each c[i,j] we can say how it was acquired:



For example, here

c[i,j] = c[i-1,j-1] +1 = 2+1=3

How to find actual LCS - continued

- Remember that

$$c[i,j] = \begin{cases} c[i-1,j-1]+1 & \text{if } x[i]=y[j], \\ \max(c[i,j-1],c[i-1,j]) & \text{otherwise} \end{cases}$$

So we can start from $c[m,n]$ and go backwards

Whenever $c[i,j] = c[i-1, j-1]+1$, remember $x[i]$ (because $x[i]$ is a part of LCS)

When i=0 or j=0 (i.e. we reached the beginning), output remembered letters in reverse order

Finding LCS

i	j	0	1	2	3	4	5
		Yj	B	D	C	A	B
0	Xi	0	0	0	0	0	0
1	A	0	0	0	0	1	1
2	B	0	1	1	1	1	2
3	C	0	1	1	2	2	2
4	B	0	1	1	2	2	3

Finding LCS (2)

i	j	0	1	2	3	4	5
		Yj	B	D	C	A	B
0	Xi	0	0	0	0	0	0
1	A	0	0	0	0	1	1
2	B	0	1	1	1	1	2
3	C	0	1	1	2	2	2
4	B	0	1	1	2	2	3

LCS (reversed order): **B C B**

LCS (straight order): **B C B**

(this string turned out to be a palindrome)