Topological Sort
Topological Sort

Want to “sort” a directed acyclic graph (DAG).

Think of original DAG as a **partial order**.

Want a **total order** that extends this partial order.
Topological Sort

- Performed on a **DAG**.
- Linear ordering of the vertices of $G$ such that if $(u, v) \in E$, then $u$ appears somewhere before $v$.

**Topological-Sort ($G$)**

1. call DFS($G$) to compute finishing times $f[v]$ for all $v \in V$
2. as each vertex is finished, insert it onto the front of a linked list
3. return the linked list of vertices

**Time:** $\Theta(V + E)$.

**Example:** On board.
Example

(Courtesy of Prof. Jim Anderson)

Linked List:
Example

Linked List:
Example

Linked List:
Example

Linked List:
Example

Linked List:
Example

Linked List:

A → B
C → B
D → E

B → 5/6
C → 6/5

D → 1/4
E → 2/3
Example

Linked List:

A → B → D
C → E

B → C

D → E

5/6
1/4
2/3
6/7

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Example

Linked List:
Example

Linked List:

B → 6/7 → 1/4 → 2/3
Example

Linked List:

A → B → C → D → E
Correctness Proof

• Just need to show if \((u, v) \in E\), then \(f[v] < f[u]\).
• When we explore \((u, v)\), what are the colors of \(u\) and \(v\)?
  – \(u\) is gray.
  – Is \(v\) gray, too?
    ➢ No, because then \(v\) would be ancestor of \(u\).
    ➢ \(\Rightarrow (u, v)\) is a back edge.
    ➢ \(\Rightarrow\) contradiction of Lemma 22.11 (DAG has no back edges).
  – Is \(v\) white?
    ➢ Then becomes descendant of \(u\).
    ➢ By parenthesis theorem, \(d[u] < d[v] < f[v] \leq f[u]\).
  – Is \(v\) black?
    ➢ Then \(v\) is already finished.
    ➢ Since we’re exploring \((u, v)\), we have not yet finished \(u\).
    ➢ Therefore, \(f[v] < f[u]\).
Strongly Connected Components

- $G$ is strongly connected if every pair $(u, v)$ of vertices in $G$ is reachable from one another.
- A **strongly connected component** (SCC) of $G$ is a maximal set of vertices $C \subseteq V$ such that for all $u, v \in C$, both $u \sim v$ and $v \sim u$ exist.
Component Graph

- $G^{SCC} = (V^{SCC}, E^{SCC})$.
- $V^{SCC}$ has one vertex for each SCC in $G$.
- $E^{SCC}$ has an edge if there’s an edge between the corresponding SCC’s in $G$.
- $G^{SCC}$ for the example considered:
$G^{SCC}$ is a DAG

**Lemma 22.13**

Let $C$ and $C'$ be distinct SCC's in $G$, let $u, v \in C$, $u', v' \in C$, and suppose there is a path $u \sim u'$ in $G$. Then there cannot also be a path $v' \sim v$ in $G$.

**Proof:**

- Suppose there is a path $v' \sim v$ in $G$.
- Then there are paths $u \sim u' \sim v'$ and $v' \sim v \sim u$ in $G$.
- Therefore, $u$ and $v'$ are reachable from each other, so they are not in separate SCC's.
Transpose of a Directed Graph

• $G^T = \text{transpose}$ of directed $G$.
  
  – $G^T = (V, E^T)$, $E^T = \{(u, v) : (v, u) \in E\}$.
  
  – $G^T$ is $G$ with all edges reversed.

• Can create $G^T$ in $\Theta(V + E)$ time if using adjacency lists.

• $G$ and $G^T$ have the same SCC’s. ($u$ and $v$ are reachable from each other in $G$ if and only if reachable from each other in $G^T$.)

Algorithm to determine SCCs

\[ \textbf{SCC}(G) \]

1. call DFS\((G)\) to compute finishing times \(f[u]\) for all \(u\)
2. compute \(G^T\)
3. call DFS\((G^T)\), but in the main loop, consider vertices in order of decreasing \(f[u]\) (as computed in first DFS)
4. output the vertices in each tree of the depth-first forest formed in second DFS as a separate SCC

\textbf{Time:} \(\Theta(V + E)\).

\textbf{Example:} On board.
Example

(Courtesy of Prof. Jim Anderson)

G

a

13/14

b

11/16

c

1/10

d

8/9

e

12/15

f

3/4

g

2/7

h

5/6

(Courtesy of Prof. Jim Anderson)
Example

$G^T$
Example

\[ \text{abe} \longrightarrow \text{cd} \]
\[ \text{fg} \rightarrow \text{h} \]
Example (2)
Example (2) DFS
Example (2) $G^T$
Example (2) DFT in $G^T$
Example (2) SCC
How does it work?

**Idea:**
- By considering vertices in second DFS in decreasing order of finishing times from first DFS, we are visiting vertices of the component graph in topologically sorted order.
- Because we are running DFS on $G^T$, we will not be visiting any $v$ from a $u$, where $v$ and $u$ are in different components.

**Notation:**
- $d[u]$ and $f[u]$ always refer to first DFS.
- Extend notation for $d$ and $f$ to sets of vertices $U \subseteq V$:
  - $d(U) = \min_{u \in U} \{d[u]\}$ (earliest discovery time)
  - $f(U) = \max_{u \in U} \{f[u]\}$ (latest finishing time)
SCCs and DFS finishing times

**Lemma 22.14**
Let $C$ and $C'$ be distinct SCC’s in $G = (V, E)$. Suppose there is an edge $(u, v) \in E$ such that $u \in C$ and $v \in C'$. Then $f(C) > f(C')$.

**Proof:**
- **Case 1: $d(C) < d(C')$**
  - Let $x$ be the first vertex discovered in $C$.
  - At time $d[x]$, all vertices in $C$ and $C'$ are white. Thus, there exist paths of white vertices from $x$ to all vertices in $C$ and $C'$.
  - By the white-path theorem, all vertices in $C$ and $C'$ are descendants of $x$ in depth-first tree.
  - By the parenthesis theorem, $f[x] = f(C) > f(C')$. 
**SCCs and DFS finishing times**

**Lemma 22.14**
Let $C$ and $C'$ be distinct SCC’s in $G = (V, E)$. Suppose there is an edge $(u, v) \in E$ such that $u \in C$ and $v \in C'$. Then $f(C) > f(C')$.

**Proof:**

- **Case 2: $d(C) > d(C')$**
  - Let $y$ be the first vertex discovered in $C'$.
  - At time $d[y]$, all vertices in $C'$ are white and there is a white path from $y$ to each vertex in $C' \Rightarrow$ all vertices in $C'$ become descendants of $y$. Again, $f[y] = f(C')$.
  - At time $d[y]$, all vertices in $C$ are also white.
  - By earlier lemma, since there is an edge $(u, v)$, we cannot have a path from $C'$ to $C$.
  - So no vertex in $C$ is reachable from $y$.
  - Therefore, at time $f[y]$, all vertices in $C$ are still white.
  - Therefore, for all $w \in C, f[w] > f[y]$, which implies that $f(C) > f(C')$. 
SCCs and DFS finishing times

Corollary 22.15
Let $C$ and $C'$ be distinct SCC’s in $G = (V, E)$. Suppose there is an edge $(u, v) \in E^T$, where $u \in C$ and $v \in C'$. Then $f(C) < f(C')$.

Proof:

• $(u, v) \in E^T \Rightarrow (v, u) \in E$.
• Since SCC’s of $G$ and $G^T$ are the same, $f(C') > f(C)$, by Lemma 22.14.
Correctness of SCC

- When we do the second DFS, on $G^T$, start with SCC $C$ such that $f(C)$ is maximum.
  - The second DFS starts from some $x \in C$, and it visits all vertices in $C$.
  - Corollary 22.15 says that since $f(C) > f(C')$ for all $C \neq C'$, there are no edges from $C$ to $C'$ in $G^T$.
  - Therefore, DFS will visit only vertices in $C$.
  - Which means that the depth-first tree rooted at $x$ contains exactly the vertices of $C$. 

Correctness of SCC

- The next root chosen in the second DFS is in SCC $C'$ such that $f(C')$ is maximum over all SCC’s other than $C$.
  - DFS visits all vertices in $C'$, but the only edges out of $C'$ go to $C$, which we’ve already visited.
  - Therefore, the only tree edges will be to vertices in $C'$.
- We can continue the process.
- Each time we choose a root for the second DFS, it can reach only
  - vertices in its SCC—get tree edges to these,
  - vertices in SCC’s already visited in second DFS—get no tree edges to these.