Design and Analysis of Algorithms

CSE 5311

Lecture 9 Median and Order Statistics

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Medians and Order Statistics

- The ith order statistic of n elements $S=\{a_1, a_2,\ldots, a_n\}$: ith smallest elements
- Also called selection problem
- Minimum and maximum
- Median, lower median, upper median
- Selection in expected/average linear time
- Selection in worst-case linear time

Order Statistics

- **The *i*th *order statistic* in a set of *n* elements is the *i*th smallest element**
- **The *minimum* is thus the 1st order statistic**
- **The *maximum* is (duh) the *n*th order statistic**
- **The *median* is the *n*/2 order statistic**
  - If *n* is even, there are 2 medians
- ***How can we calculate order statistics?***
- ***What is the running time?***

Order Statistics

- ***How many comparisons are needed to find the minimum element in a set? The maximum?***
  - To compute the maximum n − 1 comparisons are necessary and sufficient.
  - The algorithm is optimal with respect to the number of comparisons performed
  - The same is true for the minimum.

```
MINIMUM(A, n)
    min ← A[1]
    for i ← 2 to n
        do if min > A[i]
                then min ← A[i]
    return min
```

Can we find the minimum and maximum with less cost?

Yes:

— Walk through elements by pairs
— Compare each element in pair to the other
— Compare the largest to maximum, smallest to minimum

Order Statistics

- **Simultaneous computation of max and min**
  - Maintain the variables min and max . Process the n numbers in pairs.
  - For the first pair, set min to the smaller and max to the other. After that, for each new pair, compare the smaller with min and the larger with max .
  - Can be done in $3(n-3)/2$ steps

MAX-AND-MIN(A, n)

```
max ← A[n]; min ← A[n]
for i ← 1 to n/2 do
    if A[2i − 1] ≥ A[2i] then
    {  if A[2i − 1] > max then
          max ← A[2i − 1]
       if A[2i] < min then
          min ← A[2i] }
    else { if A[2i] > max then
          max ← A[2i]
       if A[2i − 1] < min then
          min ← A[2i − 1] }
return max and min
```

Example: Simultaneous Max, Min

- $n = 5$ (odd), array $A = \{2, 7, 1, 3, 4\}$

1. Set **min** = **max** = 2

2. Compare elements in pairs:

   – $1 < 7 \Rightarrow$ compare 1 with **min** and 7 with **max**

     $\Rightarrow$ **min** = 1, **max** = 7

   – $3 < 4 \Rightarrow$ compare 3 with **min** and 4 with **max**

     $\Rightarrow$ **min** = 1, **max** = 7

Total cost: $3(n-1)/2 = 6$ comparisons

Example: Simultaneous Max, Min

- $n = 6$ (even), array $A = \{2, 5, 3, 7, 1, 4\}$

1. Compare 2 with 5: $2 < 5$
2. Set **min =** 2, **max** = 5
3. Compare elements in pairs:
   - $3 < 7 \Rightarrow$ compare 3 with **min** and 7 with **max**

     $\Rightarrow$ **min** = 2, **max** = 7
   - $1 < 4 \Rightarrow$ compare 1 with **min** and 4 with **max**

     $\Rightarrow$ **min** = 1, **max** = 7

Total cost: $3n/2-2 = 7$ comparisor

$O(n\lg n)$ Algorithm

- **Suppose n elements are sorted by an $O(n\lg n)$ algorithm, e.g., MERGE-SORT**
  - Minimum: the first element; Maximum: the last element
  - The ith order statistic: the ith element.
  - Median:
    - If n is odd, then $((n+1)/2)$th element.
    - If n is even,
      - then $(\lfloor (n+1)/2 \rfloor)$th element, lower median
      - then $(\lceil (n+1)/2 \rceil)$th element, upper median
- **All selections can be done in O(1), so total: $O(n\lg n)$.**
  - Selection is a trivial problem if the input numbers are sorted.
  - But using a sorting is more like using a cannon to shoot a fly since only one number needs to computed.
- **Can we do better?**

Selection in Expected Linear Time $O(n)$

- Select ith element
- A divide-and-conquer algorithm RANDOMIZED-SELECT
- Similar to quicksort, partition the input array recursively
- Unlike quicksort, which works on both sides of the partition, just work on one side of the partition.
  - **Called prune-and-search**, prune one side, just search the other side).

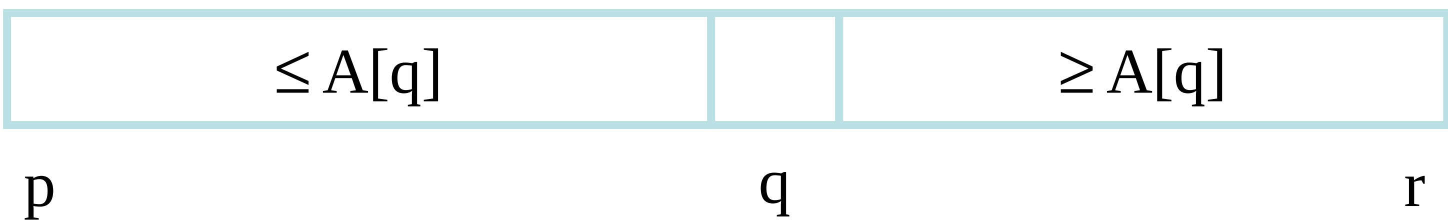
Finding Order Statistics: The Selection Problem

- **A more interesting problem is *selection*: finding the *i*th smallest element of a set**
- **We will show:**
  - A practical randomized algorithm with O(n) expected running time
  - A cool algorithm of theoretical interest only with O(n) worst-case running time

Randomized Selection

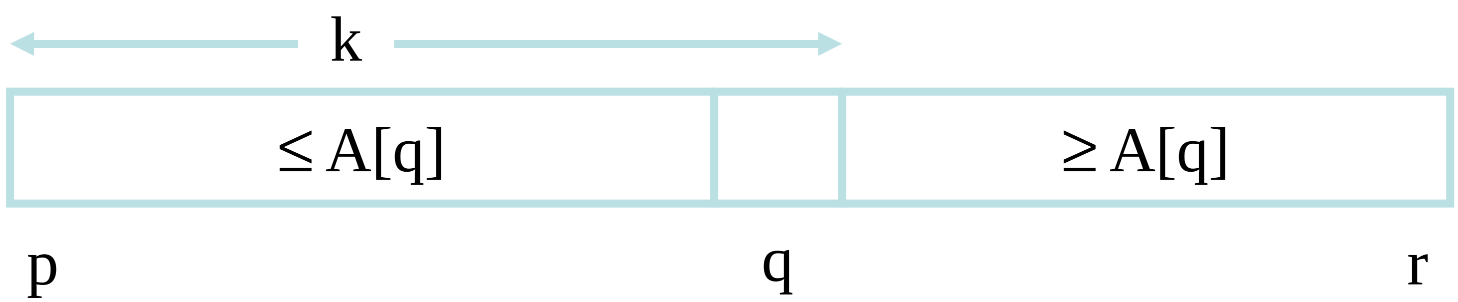
- **Key idea: use partition() from quicksort**
  - But, only need to examine one subarray
  - This savings shows up in running time: O(n)
- **We will again use a slightly different partition than the book:**

  q = RandomizedPartition(A, p, r)



Randomized Selection

```
RandomizedSelect(A, p, r, i)
    if (p == r) then return A[p];
    q = RandomizedPartition(A, p, r)
    k = q - p + 1;
    if (i == k) then return A[q];
    if (i < k) then
        return RandomizedSelect(A, p, q-1, i);
    else
        return RandomizedSelect(A, q+1, r, i-k);
```



Randomized Selection

- Analyzing **RandomizedSelect()**
  - Worst case: partition always 0:n-1

    $T(n) = T(n-1) + O(n)$ ***= ???***

    $= O(n^2)$ (arithmetic series)

    ➢ No better than sorting!
  - "Best" case: suppose a 9:1 partition

    $T(n) = T(9n/10) + O(n)$ ***= ???***

    $= O(n)$ (Master Theorem, case 3)

    ➢ Better than sorting!

    ➢ ***What if this had been a 99:1 split?***

Randomized Selection

- **Average case**
  - For upper bound, assume *i*th element always falls in larger side of partition:

$$T(n) \le \frac{1}{n}\sum_{k=0}^{n-1} T(\max(k-1, n-k)) + \Theta(n)$$

What happened here?

$$\le \frac{2}{n}\sum_{k=n/2}^{n-1} T(k) + \Theta(n)$$

  - Let's show that T(*n*) = O(*n*) by substitution

Max(k-1, n-k)=k-1 if k> ┌n/2┐
Max(k-1, n-k)=n-k if k<= ┌n/2┐

Randomized Selection

- **Assume T(n) ≤ cn for sufficiently large c:**

$$
\begin{aligned}
T(n) &\le \frac{2}{n}\sum_{k=n/2}^{n-1} T(k) + \Theta(n) && \textbf{The recurrence we started with} \\
&\le \frac{2}{n}\sum_{k=n/2}^{n-1} ck + \Theta(n) && \textbf{Substitute } T(n) \le cn \textbf{ for } T(k) \\
&= \frac{2c}{n}\left(\sum_{k=1}^{n-1} k - \sum_{k=1}^{n/2-1} k\right) + \Theta(n) && \textbf{“Split” the recurrence} \\
&= \frac{2c}{n}\left(\frac{1}{2}(n-1)n - \frac{1}{2}\left(\frac{n}{2}-1\right)\frac{n}{2}\right) + \Theta(n) && \textbf{Expand arithmetic series} \\
&= c(n-1) - \frac{c}{2}\left(\frac{n}{2}-1\right) + \Theta(n) && \textbf{Multiply it out}
\end{aligned}
$$

Randomized Selection

- Assume T(n) ≤ cn for sufficiently large c:

$$
\begin{aligned}
T(n) &\leq c(n-1) - \frac{c}{2}\left(\frac{n}{2} - 1\right) + \Theta(n) && \textbf{The recurrence so far} \\
&= cn - c - \frac{cn}{4} + \frac{c}{2} + \Theta(n) && \textbf{Multiply it out} \\
&= cn - \frac{cn}{4} - \frac{c}{2} + \Theta(n) && \textbf{Subtract c/2} \\
&= cn - \left(\frac{cn}{4} + \frac{c}{2} - \Theta(n)\right) && \textbf{Rearrange the arithmetic} \\
&\leq cn \quad \text{(if c is big enough)} && \textbf{What we set out to prove}
\end{aligned}
$$

Worst-Case Linear-Time Selection

- **Randomized algorithm works well in practice**
- **What follows is a worst-case linear time algorithm, really of theoretical interest only**
- **Basic idea:**
  - Generate a good partitioning element
  - Call this element x

Worst-Case Linear-Time Selection

- **The algorithm in words:**
  1. Divide the elements into groups of five, where the last group may have less than five elements in case when the input array size is not a multiple of five.
  2. Find median of each group (*How? How long?*). Ties can be broken arbitrarily
  3. Make a recursive call **Select()** to calculate the median of the medians. Set x to the median.
  4. Partition the n elements around x. Let k = rank(x)
  5. **if** (i == k) **then** return x

     if (i < k) **then** use Select() recursively to find ith smallest element in first partition
     else (i > k) use Select() recursively to find (i-k)th smallest element in last partition

k = rank(x), x is the k-th smallest element and there are n-k elements on the high side of the partition

Example

- Find the –11th smallest element in array:

  A = {12, 34, 0, 3, 22, 4, 17, 32, 3, 28, 43, 82, 25, 27, 34, 2 ,19 ,12 ,5 ,18 ,20 ,33, 16, 33, 21, 30, 3, 47}

1. Divide the array into groups of 5 elements

12	4	43	2	20	30
34	17	82	19	33	3
0	32	25	12	16	47
3	3	27	5	33	
22	28	34	18	21	

Example

2. Sort the groups and find their medians

0	4	25	2	20	3
3	3	27	5	16	30
12	17	34	12	21	47
34	32	43	19	33	
22	28	82	18	33	

3. Find the median of the medians

12, 12, 17, 21, 34, 30

Example

4. Partition the array around the median of medians (17)

   First partition:

   {12, 0, 3, 4, 3, 2, 12, 5, 16, 3}

   Pivot:

   17 (position of the pivot is q = 11)

   Second partition:

   {34, 22, 32, 28, 43, 82, 25, 27, 34, 19, 18, 20, 33, 33, 21, 30, 47}

To find the 6-th smallest element we would have to recurse our search in the first partition.

Analysis of Running Time

- Step 1: making groups of 5 elements takes **O(n)**
- Step 2: sorting n/5 groups in O(1) time each takes **O(n)**
- Step 3: calling SELECT on $\lceil n/5 \rceil$ medians takes time **T($\lceil$n/5$\rceil$)**
- Step 4: partitioning the n-element array around x takes **O(n)**
- Step 5: recursion on one partition takes

  depends on the size of the partition!!

Worst-Case Linear-Time Selection

- **(Sketch situation on the board)**
- ***How many of the 5-element medians are ≤ x?***
  - At least 1/2 of the medians = $\lfloor \lfloor n/5 \rfloor / 2 \rfloor = \lfloor n/10 \rfloor$
- ***How many elements are ≤ x?***
  - At least $3 \lfloor n/10 \rfloor$ elements
- **For large *n*,** $3 \lfloor n/10 \rfloor \geq n/4$ ***(How large?)***
- **So at least $n/4$ elements ≤ *x***
- **Similarly: at least $n/4$ elements ≥ *x***

Worst-Case Linear-Time Selection

- Thus after partitioning around x, step 5 will call Select() on at most $3n/4$ elements
- The recurrence is therefore:

$$
\begin{aligned}
T(n) &\le T(\lfloor n/5 \rfloor) + T(3n/4) + \Theta(n) \\
&\le T(n/5) + T(3n/4) + \Theta(n) && \lfloor n/5 \rfloor \le n/5 \\
&\le cn/5 + 3cn/4 + \Theta(n) && \text{Substitute } T(n) = cn \\
&= 19cn/20 + \Theta(n) && \text{Combine fractions} \\
&= cn - (cn/20 - \Theta(n)) && \text{Express in desired form} \\
&\le cn \quad \text{if } c \text{ is big enough} && \text{What we set out to prove}
\end{aligned}
$$

Worst-Case Linear-Time Selection

- **Intuitively:**
  - Work at each level is a constant fraction (19/20) smaller
    - ➢ Geometric progression!
  - Thus the O(n) work at the root dominates

Linear-Time Median Selection

- **Given a "black box" O(n) median algorithm, what can we do?**
  - *i*th order statistic:
    - Find median x
    - Partition input around x
    - if ($i \leq (n+1)/2$) recursively find *i*th element of first half
    - else find (i - $(n+1)/2$)th element in second half
    - $T(n) = T(n/2) + O(n) = O(n)$
  - ***Can you think of an application to sorting?***

Linear-Time Median Selection

- **Worst-case O(n lg n) quicksort**
  - Find median *x* and partition around it
  - Recursively quicksort two halves
  - $T(n) = 2T(n/2) + O(n) = O(n \lg n)$

Summary

- **The *i*th order statistic of *n* elements S={a_1, a_2,…, a_n} : *i*th smallest elements:**
  - Minimum and maximum.
  - Median, lower median, upper median
- **Selection in expected/average linear time**
  - Worst case running time
  - Prune-and-search
- **Selection in worst-case linear time:**