CSE 4345 / CSE 5315 - *Computational Methods*

Homework 2- Sample Solution - Fall 2011

Due Date: Oct. 7 2011

Problems marked with * are required only for students of CSE 5315 but will be graded for extra credit for students of CSE 4345.

Solving Equations

1. Fix Point Methods
   Consider the following instances of the root finding problem, $f(x^*) = 0$.

   - $f(x) = x^3 - 2x^2 + 7$
   - $f(x) = sin(x) * cos(x)$
   - $f(x) = e^{x-3} - x^2$

   a) For each of the root finding problems, derive two corresponding fixed point functions, $x^* = g(x^*)$.

      - $f(x) = x^3 - 2x^2 + 7$
        - $g(x) = \frac{1}{2}x - \frac{7}{2x^2}$
        - $g(x) = \sqrt[3]{2x^2 - 7}$
      - $f(x) = sin(x) * cos(x)$
        - $g(x) = arcsin(sin^2(x))$
        - $g(x) = x + sin(x) * cos(x)$
      - $f(x) = e^{x-3} - x^2$
        - $g(x) = \sqrt{e^{x-3}}$
        - $g(x) = 2\ln x + 3$

   b) For each of your functions determine whether fixed point iteration in a neighborhood of a root would converge (you can pick one specific root for functions that have multiple roots).

      - $f(x) = x^3 - 2x^2 + 7$
        - $g(x) = \frac{1}{2}x - \frac{7}{2x^2}$: Root is ≈ -1.429
          Derivative : $g'(x) = \frac{1}{2} + \frac{7}{x^3}$ at root is -1.899 and will therefore not converge.
        - $g(x) = \sqrt[3]{2x^2 - 7}$: Root is ≈ -1.429
          Derivative : $g'(x) = \frac{1}{3}\left(2x^2 - 7\right)^{-\frac{2}{3}} 4x$ at root is 0.9335 and will therefore converge.

- $f(x) = sin(x) * cos(x)$
  - $g(x) = arcsin(sin^2(x))$: Root is 0
    Derivative : $g'(x) = \frac{1}{\sqrt{1-sin^4(x)}} 2sin(x)cos(x)$ at root is 0 and will therefore converge.
  - $g(x) = x + sin(x) * cos(x)$: Root is 0
    Derivative : $g'(x) = 1 + sin(x)(-sin(x)) + cos(x)cos(x)$ at root is 2 and will therefore not converge.
- $f(x) = e^{x-3} - x^2$
  - $g(x) = \sqrt{e^{x-3}}$: Root is ≈ 6.848
    Derivative : $g'(x) = \frac{1}{2\sqrt{e^{x-3}}} e^{x-3} = \frac{1}{2}\sqrt{e^{x-3}}$ at root is 3.424 and will therefore not converge.
  - $g(x) = 2 \ln x + 3$: Root is ≈ 6.848
    Derivative : $g'(x) = \frac{2}{x}$ at root is 0.292 and will therefore converge.

2. Bisection and Newton's Method

   a) Implement the Bisection Method and the Secant Method for root finding. For the Bisection Method you do not have to implement a method to determine the initial bracket (you can enter it manually).

   b) Consider the following (extremely simplified) computer network scenario where a set of computers are simultaneously streaming content across a single network connection. Furthermore, assume that that all computers are synchronized to data slots and data packets of 1024 bytes are transmitted at a time. The network has a bandwidth of $b\ kilobytes/s$ ($b*1024$ bytes/s) and each of the computers streams data asynchronously at an average rate of $m\ kilobytes/s$, implying that each computer has a probability of $\frac{m}{b}$ to attempt to transmit a packet in each data slot. If multiple computers attempt to send packets in the same time slot, a collision occurs and the packets are lost, leading to a loss of k data packets per second (and thus to a decrease in the quality of the video the customer is watching).

   Determine a system equation that characterizes the the expected packet loss per second, k, for each stream as a function of the connection bandwidth, b, the data rate of each stream, m, and the number of streams (computers), n, and determine the maximum number of streams that the network can support for the following bandwidth parameters and tolerable packet loss rates. To solve for this number, use both, Interval Bisection and the Secant method and provide, in addition to the solution, what your termination criterion was and how many iterations each of the algorithms required for each of the conditions.

   - $b = 10^6$, $m = 5 * 10^4$, $k \leq 500$
   - $b = 10^7$, $m = 10^5$, $k \leq 1000$
   - $b = 10^7$, $m = 3 * 10^4$, $k \leq 1000$

   The equation can be derived by observing that a collision happens every time two packets are sent in the same slot. Thus the likelihood that a packet from a particular machine does not collide is equal to the likelihood that none of the other $n-1$ computers transmit in

the same slot which is equal to $\left(1-\frac{m}{b}\right)^{n-1}$. Therefore the expected loss for any stream is $k = m\left(1-\left(1-\frac{m}{b}\right)^{n-1}\right) = m - m\left(1-\frac{m}{b}\right)^{n-1}$.
Treating this as a function of n, this can be rewritten into a root finding problem $f(n) = (m-k) - m\left(1-\frac{m}{b}\right)^{n-1} = 0$;
The solutions for this for the above cases are 1, 2, and 12, respectively.

c)* Apply both methods to the following functions and observe their convergence by noting the remaining error between the final root and the approximation for each iteration of the two algorithms. Indicate which algorithm converges faster for the given problem.

- $f(x) = 3 - 4x + 7x^2 - x^3$
- $f(x) = x - 4sin(x) + 1$
- $f(x) = \frac{cos(x)-x}{\sqrt{x^2+1}}$

Systems of Linear Equations

4. LU Factorization with Partial Pivoting

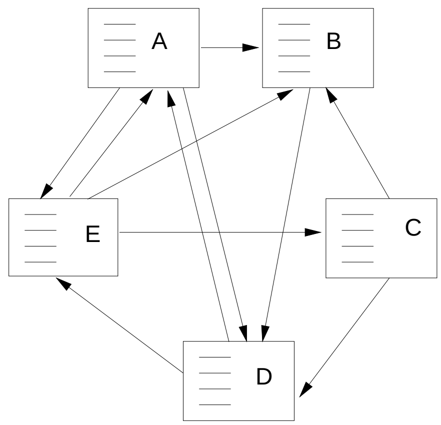
a) Implement LU Factorization for a linear system with 5 equations with 5 variables. Evaluate your implementation on three different systems of linear equations of your choice.

b) Use LU Factorization to compute Page Rank for a set of web pages
Google's search engine is originally built around the concept of Page Rank which assigns each web page an importance value based on the importance of web pages that link to it. For example, consider 3 web pages, A, B, and C where A links to B, B links to A and C, and C links to B and A. Further assume that each of the pages have a rank of r_A, r_B, r_C, respectively and that a page's popularity (rank) distributes evenly among the pages it links to. Then the relations between the pages can be represented by the following linking matrix L where each row represents the importance the corresponding page gets assigned by the links from other pages and the columns correspond to the way in which a web page distributes its own Page Rank to the other pages:

$$L = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{pmatrix}$$

Given this matrix, the rank of each page can be represented by $L\vec{r} = \vec{r}$, or alternatively, $(L-I)\vec{r} = 0$, and thus as an instance of the root finding problem for a system of linear equations. Note, however, that this system is inherently ill conditioned since the entire system has an infinite number of solutions due to the fact that scaling $\vec{r}$ does not have any influence on the solution of the system (in particular in this system the last equation is always equal to the negative sum of the previous $n-1$ equations since the columns are normalized and each column in $L - I$ thus sums to 0). One way to resolve this is to add equations that constrain the values of $\vec{r}$ such as $\sum_i r_i = 1$. While the system now has more equations than the n variables, we can still apply the solution methods using partial (or complete) pivoting to derive a solution by simply using only the top n rows in the matrix after elimination (all others will be all 0s). If the rank of the matrix is below n this implies we can assign arbitrary values to some of the parameters.

For this question consider the following link graph describing 5 web pages where arrows indicate links where one page refers to another.



Determine the link matrix L for this graph. Using this, establish the system of equations described by $(L - I)\vec{r} = 0$. Replace the last equation in this system with the one requiring the sum or importance values to be equal to 1 and determine a set of consistent Page ranks using your equation solver.

The link matrix is given by:

$$L = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{3} & 0 & \frac{1}{2} & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & \frac{1}{3} \\ \frac{1}{3} & 1 & \frac{1}{2} & 0 & 0 \\ \frac{1}{3} & 0 & 0 & \frac{1}{2} & 0 \end{pmatrix}$$

From this we get

$$(L - I)\vec{r} = \begin{pmatrix} -1 & 0 & 0 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{3} & -1 & \frac{1}{2} & 0 & \frac{1}{3} \\ 0 & 0 & -1 & 0 & \frac{1}{3} \\ \frac{1}{3} & 1 & \frac{1}{2} & -1 & 0 \\ \frac{1}{3} & 0 & 0 & \frac{1}{2} & -1 \end{pmatrix} = 0$$

And to address the insufficient rank of the matrix we replace the last row:
From this we get

$$\begin{pmatrix} -1 & 0 & 0 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{3} & -1 & \frac{1}{2} & 0 & \frac{1}{3} \\ 0 & 0 & -1 & 0 & \frac{1}{3} \\ \frac{1}{3} & 1 & \frac{1}{2} & -1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \vec{r} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

c)* Add partial pivoting to your implementation of LU factorization and compute the solution for the following system of equations with and without partial pivoting. Determine the computation error in each of the variables and compare the errors obtained with and without partial pivoting. (You can use MatLab - or any other system's - linear system solver to compute a solution that you can assume to be the correct one.) Also compute the conditioning number for this system of equations (you can choose to use the 1-norm or the infinity-norm for the calculation).

$$\begin{aligned} 51 * x_1 + 7 * 10^{-2} x_2 - 4 * x_3 &= 5 \\ 3 * x_1 + 10^{-14} x_2 - 2 * x_4 + x_5 &= 3 \\ 10^{-2} x_1 + 4 * x_2 - 4 * 10^{-8} x_3 - x_5 &= -1 \\ 400 * x_1 + 7 x_2 - 2 * x_4 &= 10 \\ 10^4 x_2 + 7 x_3 &= -2 \end{aligned}$$

Systems of Nonlinear Equations

To provide additional services for their customers while they are visiting the Texas fair, a cell-phone company decides to add a location feature to a special cell-phone app that uses the strengths of the signal coming from three cell phone towers around the fair grounds to determine the location. Assume that the transmitters on the three towers are located around the fairgrounds at the following locations (in x,y,z coordinates): $(0, 0, 10), (150, 160, 10), (90, -90, 10)$. Assuming the relation between the distance to the i^{th}cell tower, d_i, and the signal strength from that tower, r_i, is $r_i = \frac{10^5}{d_i^2}$ and that the elevation (i.e. the z coordinate) can vary, the strengths of the signals from the three towers can be used to determine the (x,y,z) location of the cell phone by first measuring r_i, translating this into distances to the towers, d_i, and finding a point, (x, y, z) that has the correct distances from all three towers (since we know where the towers are we know that the distance to each tower is if the cell phone is at location (x,y,z)).

5. Multivariate Newton's Method

   a) Formulate the complete system of equations for the location system with x, y, z as variables and r_i) as the function values (i.e. derive one equation for each r_i)

      Given the relation between signal strength and the distance, we can generate a set of squared distance equations:

$$
\begin{aligned}
d_1^2 &= (x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2 = x^2 + y^2 + (z - 10)^2 = \frac{10^5}{r_1} \\
d_2^2 &= (x - x_2)^2 + (y - y_2)^2 + (z - z_2)^2 = (x - 150)^2 + (y - 160)^2 + (z - 10)^2 = \frac{10^5}{r_2} \\
d_3^2 &= (x - x_3)^2 + (y - y_3)^2 + (z - z_3)^2 = (x - 90)^2 + (y + 90)^2 + (z - 10)^2 = \frac{10^5}{r_3}
\end{aligned}
$$

      From these we get our system of equations for r_i:

$$
\begin{aligned}
r_1 &= \frac{10^5}{x^2 + y^2 + (z - 10)^2} \\
r_2 &= \frac{10^5}{(x - 150)^2 + (y - 160)^2 + (z - 10)^2} \\
r_3 &= \frac{10^5}{(x - 90)^2 + (y + 90)^2 + (z - 10)^2}
\end{aligned}
$$

   b) Implement the Multivariate Newton Method to solve the system for the following set of signal strength measurements. You can use either your own or an already implemented method for matrix inversion.

To use the multivariate Newton Method we need to derive the Jacobian:

$$J_r(x) = \begin{pmatrix} -\frac{10^5 2x}{\left(x^2+y^2+(z-10)^2\right)^2} & -\frac{10^5 2y}{\left(x^2+y^2+(z-10)^2\right)^2} & -\frac{10^5 2(z-10)}{\left(x^2+y^2+(z-10)^2\right)^2} \\ -\frac{10^5 2(x-150)}{\left((x-150)^2+(y-160)^2+(z-10)^2\right)^2} & -\frac{10^5 2(y-160)}{\left((x-150)^2+(y-160)^2+(z-10)^2\right)^2} & -\frac{10^5 2(z-10)}{\left((x-150)^2+(y-160)^2+(z-10)^2\right)^2} \\ -\frac{10^5 2(x-90)}{\left((x-90)^2+(y+90)^2+(z-10)^2\right)^2} & -\frac{10^5 2(y+90)}{\left((x-90)^2+(y+90)^2+(z-10)^2\right)^2} & -\frac{10^5 2(z-10)}{\left((x-90)^2+(y+90)^2+(z-10)^2\right)^2} \end{pmatrix}$$

- $\vec{r} = (19.607843, 4.504505, 4.694836)^T$
- $\vec{r} = (3.327787, 46.511628, 46.511628)^T$