Signal & Weight Vector Spaces

Notation

Vectors in $\Re^n$.

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Generalized Vectors.

$$\mathcal{X}$$

Vector Space

1. An operation called vector addition is defined such that if $x \in X$ and $y \in X$ then $x + y \in X$.
2. $x + y = y + x$
3. $(x + y) + z = x + (y + z)$
4. There is a unique vector $0 \in X$, called the zero vector, such that $x + 0 = x$ for all $x \in X$.
5. For each vector there is a unique vector in X, to be called $(-x)$, such that $x + (-x) = 0$.

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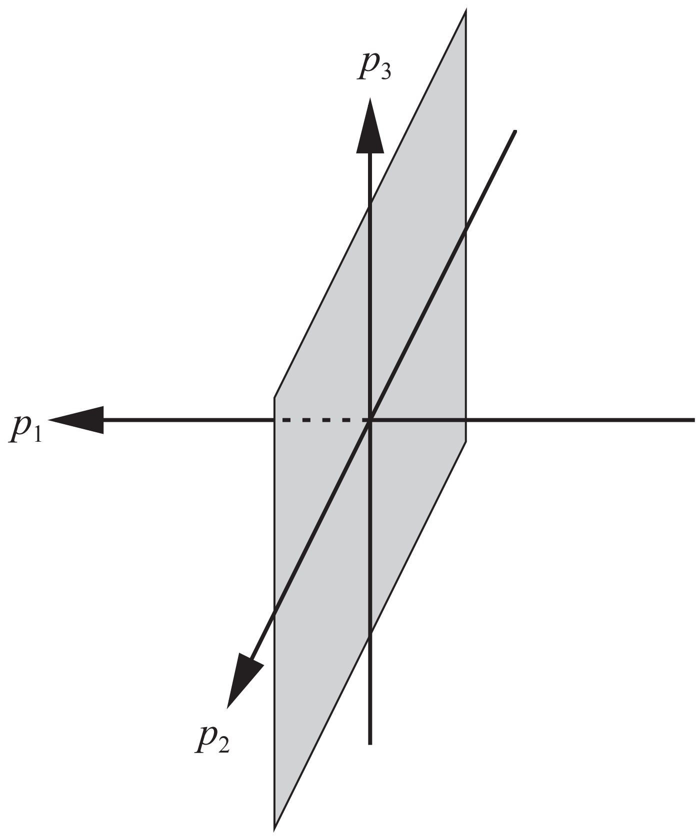
Vector Space (Cont.)

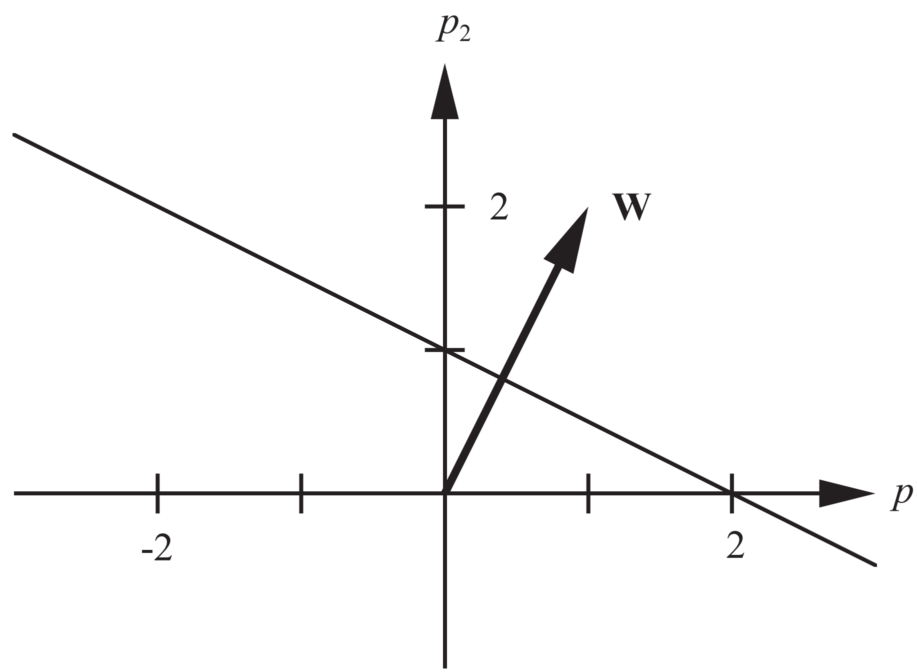
6. An operation, called multiplication, is defined such that for all scalars $a \in F$, and all vectors $x \in X$, $a\,x \in X$.
7. For any $x \in X$, $1x = x$ (for scalar 1).
8. For any two scalars $a \in F$ and $b \in F$, and any $x \in X$, $a(bx) = (ab)\,x$.
9. $(a+b)\,x = a\,x + b\,x$.
10. $a(x + y) = a\,x + a\,y$

Examples (Decision Boundaries)

Is the p_2, p_3 plane a vector space?

Is the line $p_1 + 2p_2 - 2 = 0$ a vector space?





Other Vector Spaces

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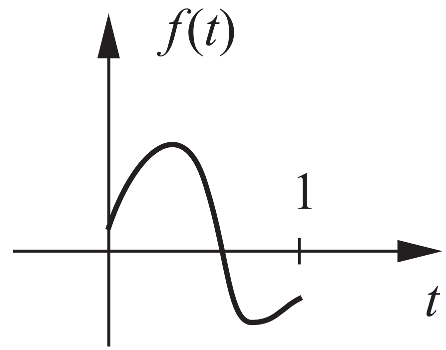
Polynomials of degree 2 or less.

$$\chi = 2 + t + 4t^2$$

$$y = 1 + 5t$$

Continuous functions in the interval [0,1].



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Linear Independence

If

$$a_1\chi_1 + a_2\chi_2 + \ldots + a_n\chi_n = 0$$

implies that each

$$a_i = 0$$

then

$$\{\chi_i\}$$

is a set of linearly independent vectors.

Example (Banana and Apple)

$$\mathbf{p}_1 = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} \qquad \mathbf{p}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

Let

$$a_1\mathbf{p}_1 + a_2\mathbf{p}_2 = \mathbf{0}$$

$$\begin{bmatrix} -a_1 + a_2 \\ a_1 + a_2 \\ -a_1 + (-a_2) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This can only be true if

$$a_1 = a_2 = 0$$

Therefore the vectors are independent.

Spanning a Space

A subset **spans** a space if every vector in the space can be written as a linear combination of the vectors in the subspace.

$$\chi = x_1 u_1 + x_2 u_2 + \ldots + x_m u_m$$

Basis Vectors

- A set of basis vectors for the space X is a set of vectors which spans X and is linearly independent.

- The dimension of a vector space, $\mathrm{Dim}(X)$, is equal to the number of vectors in the basis set.

- Let X be a finite dimensional vector space, then every basis set of X has the same number of elements.

Example

Polynomials of degree 2 or less.

Basis A:

$u_1 = 1 \qquad u_2 = t \qquad u_3 = t^2$

Basis B:

$u_1 = 1 - t \qquad u_2 = 1 + t \qquad u_3 = 1 + t + t^2$

(Any three linearly independent vectors in the space will work.)

How can you represent the vector $x = 1 + 2t$ using both basis sets?

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Inner Product / Norm

A scalar function of vectors x and y can be defined as an **inner product**, (x,y), provided the following are satisfied (for real inner products):

- $(x,y) = (y,x)$
- $(x,ay_1+by_2) = a(x,y_1)+b(x,y_2)$
- $(x,x) \geq 0$, where equality holds iff $x = 0$.

A scalar function of a vector x is called a **norm**, $||x||$, provided the following are satisfied:

- $||x|| \geq 0$.
- $||x||=0$ iff $x = 0$.
- $||ax|| = |a|\ ||x||$ for scalar a.
- $||x+ y|| \leq ||x|| + ||y||$.

Example

Standard Euclidean Inner Product

$$\mathbf{x}^T\mathbf{y} = x_1y_1 + x_2y_2 + \cdots + x_ny_n$$

Standard Euclidean Norm

$$\|x\| = (x, x)^{1/2}$$

$$\|\mathbf{x}\| = (\mathbf{x}^T\mathbf{x})^{1/2} = (x_1^{\ 2} + x_2^{\ 2} + \ldots + x_n^{\ 2})^{\ 1/2}$$

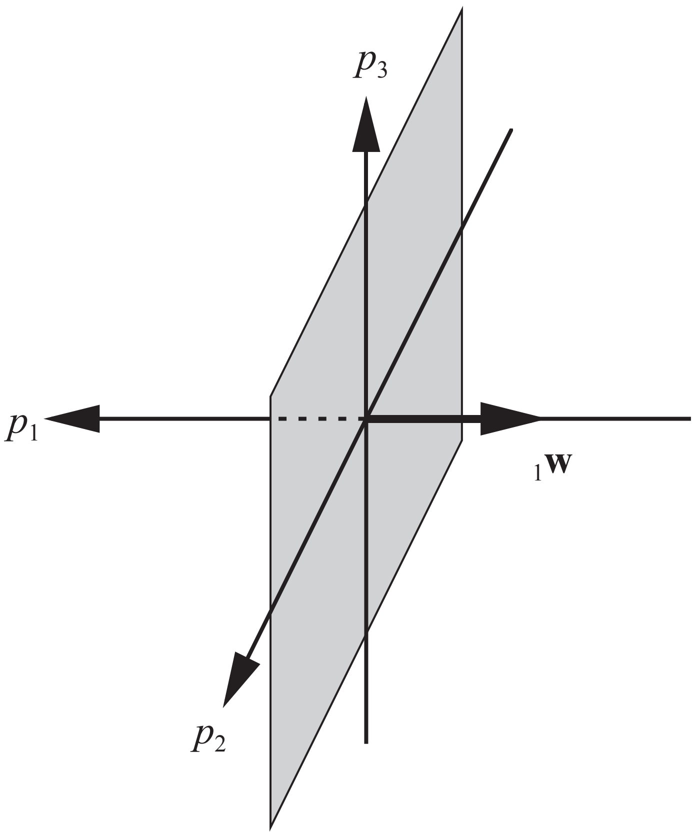
Angle

$$\cos(\theta) = (x,y)/(\|x\|\ \|y\|)$$

Orthogonality

Two vectors $\chi, y \in X$ are orthogonal if $(\chi, y) = 0$.

Example



Any vector in the p_2,p_3 plane is orthogonal to the weight vector.

5 Gram-Schmidt Orthogonalization

Independent Vectors		Orthogonal Vectors
$y_1, y_2, \ldots, y_n$	⇒	$v_1, v_2, \ldots, v_n$

Step 1: Set first orthogonal vector to first independent vector.

$$v_1 = y_1$$

Step 2: Subtract the portion of y_2 that is in the direction of v_1.

$$v_2 = y_2 - a v_1$$

Where a is chosen so that v_2 is orthogonal to v_1:

$$(v_1, v_2) = (v_1, y_2 - a v_1) = (v_1, y_2) - a(v_1, v_1) = 0$$

$$a = \frac{(v_1, y_2)}{(v_1, v_1)}$$

Gram-Schmidt (Cont.)

Projection of y_2 on v_1:

$$\frac{(v_1, y_2)}{(v_1, v_1)} v_1$$

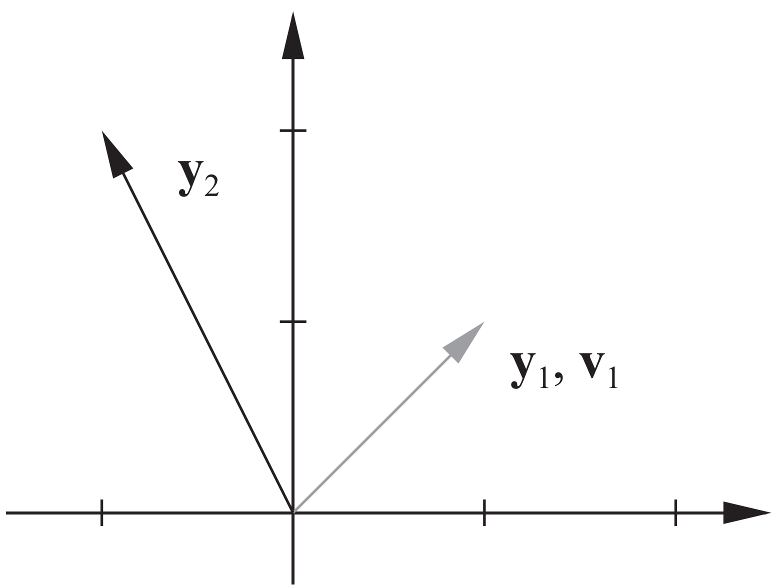
Step k: Subtract the portion of y_k that is in the direction of all previous v_i.

$$v_k = y_k - \sum_{i=1}^{k-1} \frac{(v_i, y_k)}{(v_i, v_i)} v_i$$

Example

$$\mathbf{y}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \mathbf{y}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

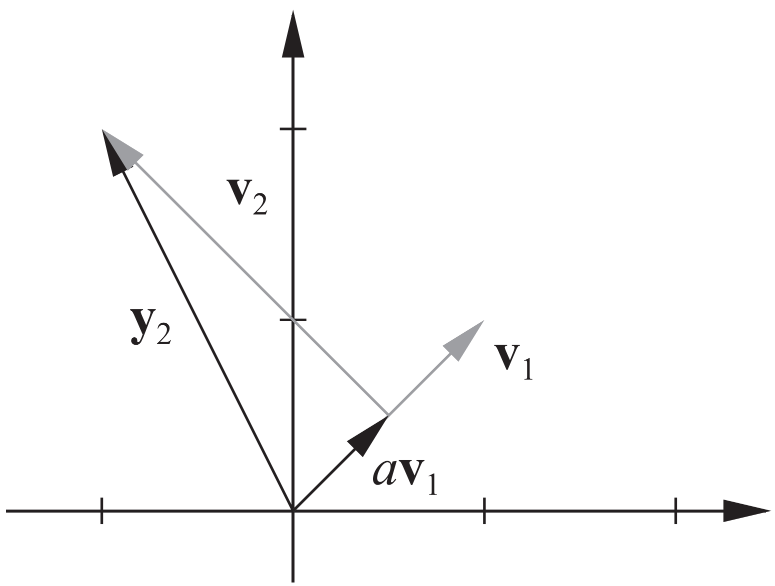
Step 1. $\mathbf{v}_1 = \mathbf{y}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$



Example (Cont.)

Step 2.

$$\mathbf{v}_2 = \mathbf{y}_2 - \frac{\mathbf{v}_1^T\mathbf{y}_2}{\mathbf{v}_1^T\mathbf{v}_1}\mathbf{v}_1 = \begin{bmatrix}-1\\2\end{bmatrix} - \frac{\begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}-1\\2\end{bmatrix}}{\begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}-1\\2\end{bmatrix} - \begin{bmatrix}0.5\\0.5\end{bmatrix} = \begin{bmatrix}-1.5\\1.5\end{bmatrix}$$



5 Vector Expansion

If a vector space X has a basis set $\{v_1, v_2, ..., v_n\}$, then any $\chi \in X$ has a unique vector expansion:

$$\chi = \sum_{i=1}^{n} x_i v_i = x_1 v_1 + x_2 v_2 + ... + x_n v_n$$

If the basis vectors are **orthogonal**, and we take the inner product of v_j and χ:

$$(v_j, \chi) = (v_j, \sum_{i=1}^{n} x_i v_i) = \sum_{i=1}^{n} x_i (v_j, v_i) = x_j (v_j, v_j)$$

Therefore the coefficients of the expansion can be computed:

$$x_j = \frac{(v_j, \chi)}{(v_j, v_j)}$$

Column of Numbers

The vector expansion provides a meaning for writing a vector as a column of numbers.

$$\chi = \sum_{i=1}^{n} x_i v_i = x_1 v_1 + x_2 v_2 + \ldots + x_n v_n$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

To interpret **x**, we need to know what basis was used for the expansion.

5 Reciprocal Basis Vectors

Definition of reciprocal basis vectors, r_i:

$$(r_i, v_j) = 0 \qquad i \neq j$$
$$= 1 \qquad i = j$$

where the basis vectors are $\{v_1, v_2, ..., v_n\}$, and
the reciprocal basis vectors are $\{r_1, r_2, ..., r_n\}$.

For vectors in $\Re^n$ we can use the following inner product:

$$(r_i, v_j) = \mathbf{r}_i^T \mathbf{v}_j$$

Therefore, the equations for the reciprocal basis vectors become:

$$\mathbf{R}^T \mathbf{B} = \mathbf{I} \quad \Longrightarrow \quad \mathbf{R}^T = \mathbf{B}^{-1}$$

$$\mathbf{B} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_n \end{bmatrix} \qquad \mathbf{R} = \begin{bmatrix} \mathbf{r}_1 & \mathbf{r}_2 & \dots & \mathbf{r}_n \end{bmatrix}$$

Vector Expansion

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$$\chi = x_1 v_1 + x_2 v_2 + \ldots + x_n v_n$$

Take the inner product of the j th reciprocal basis vector with the vector to be expanded:

$$(r_j,\chi) = (r_j, \sum_{i=1}^{n} x_i v_i) = \sum_{i=1}^{n} x_j (r_j, v_i) = x_j (r_j, v_j) = x_j$$

Because, by definition of the reciprocal basis vectors:

$$\begin{aligned} (r_i, v_j) &= 0 \qquad i \neq j \\ &= 1 \qquad i = j \end{aligned}$$

In general, we then have (even for nonorthogonal basis vectors):

$$x_j = (r_j,\chi)$$

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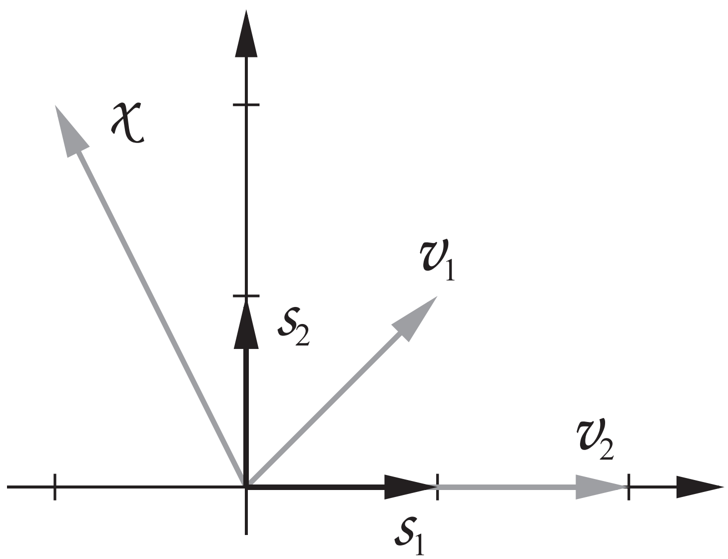
Example

Basis Vectors:

$$\mathbf{v}_1^s = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \mathbf{v}_2^s = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

Vector to Expand:

$$\mathbf{x}^s = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$



Example (Cont.)

Reciprocal Basis Vectors:

$$\mathbf{R}^T = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 \\ 0.5 & -0.5 \end{bmatrix} \qquad \mathbf{r}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \mathbf{r}_2 = \begin{bmatrix} 0.5 \\ -0.5 \end{bmatrix}$$

Expansion Coefficients:

$$x_1^v = \mathbf{r}_1^T \mathbf{x}^s = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = 2$$

$$x_2^v = \mathbf{r}_2^T \mathbf{x}^s = \begin{bmatrix} 0.5 & -0.5 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = -1.5$$

Matrix Form:

$$\mathbf{x}^v = \mathbf{R}^T \mathbf{x}^s = \mathbf{B}^{-1} \mathbf{x}^s = \begin{bmatrix} 0 & 1 \\ 0.5 & -0.5 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1.5 \end{bmatrix}$$

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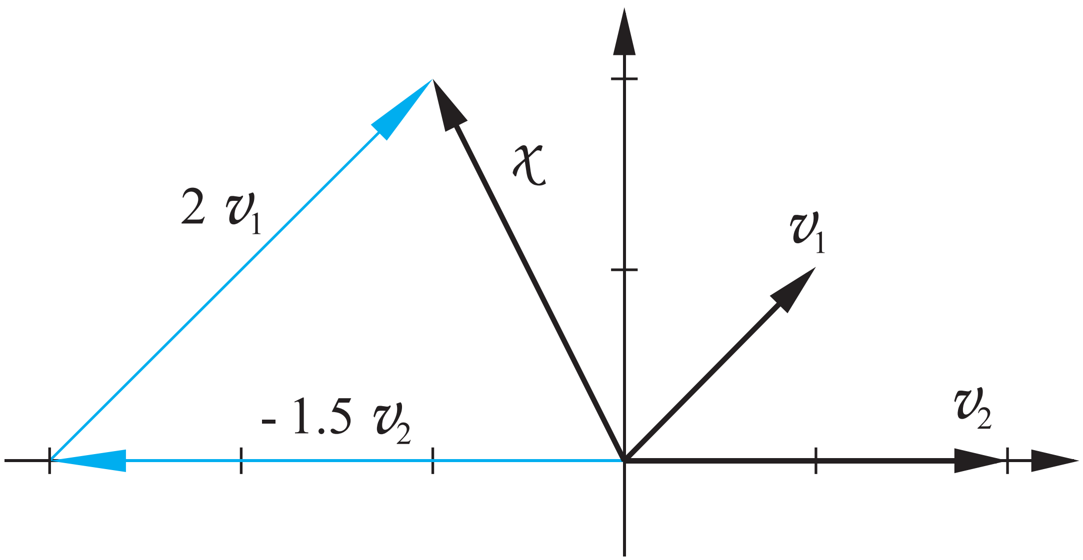
Example (Cont.)

$$\chi = (-1)s_1 + 2s_2 = 2v_1 - 1.5v_2$$



$$\mathbf{x}^s = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \qquad \mathbf{x}^v = \begin{bmatrix} 2 \\ -1.5 \end{bmatrix}$$

The interpretation of the column of numbers depends on the basis set used for the expansion.