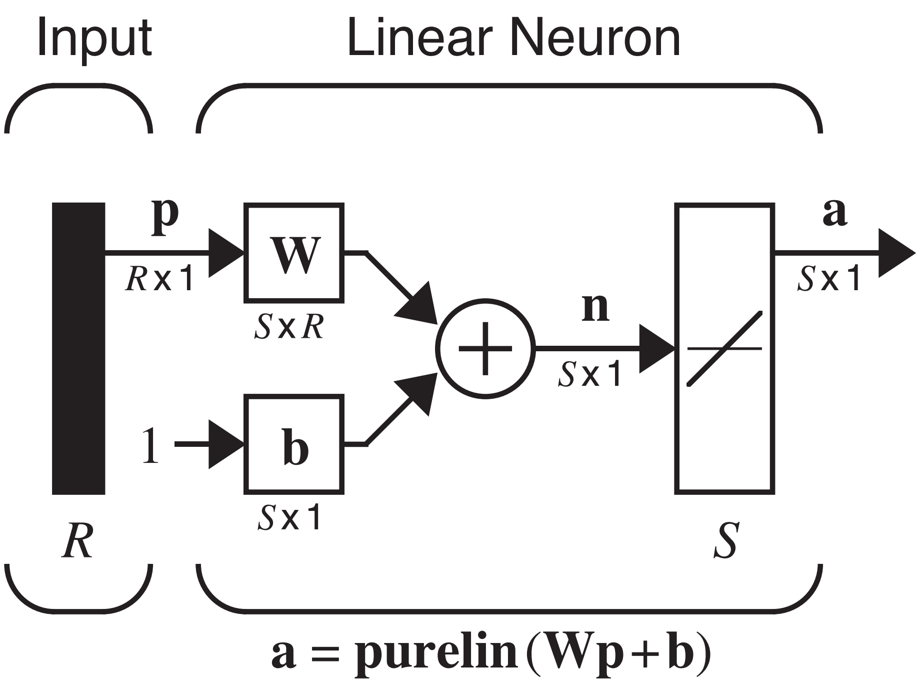
Widrow-Hoff Learning

(LMS Algorithm)

ADALINE Network

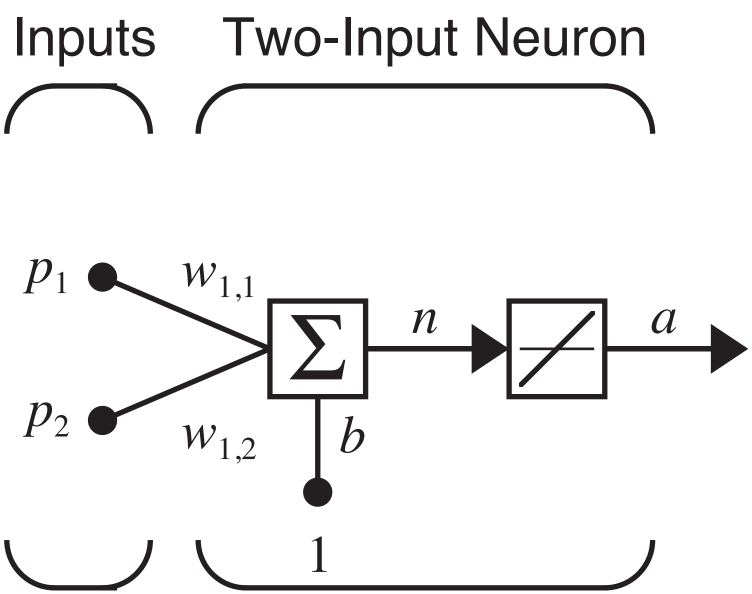


$$\mathbf{a} = \mathbf{purelin}(\mathbf{W}\mathbf{p} + \mathbf{b}) = \mathbf{W}\mathbf{p} + \mathbf{b}$$

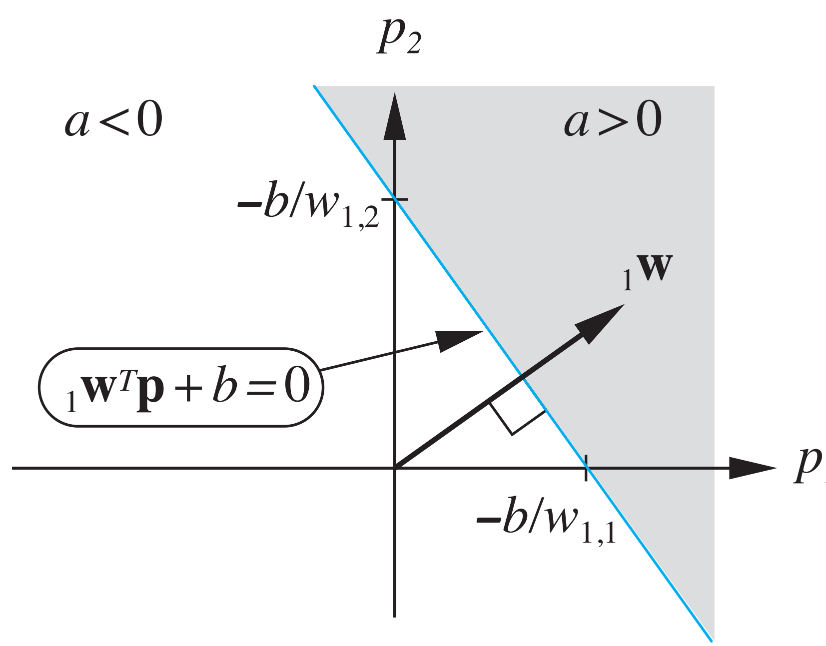
$$a_i = purelin(n_i) = purelin({}_i\mathbf{w}^T\mathbf{p} + b_i) = {}_i\mathbf{w}^T\mathbf{p} + b_i$$

$${}_i\mathbf{w} = \begin{bmatrix} w_{i,1} \\ w_{i,2} \\ \vdots \\ w_{i,R} \end{bmatrix}$$

Two-Input ADALINE



$$a = purelin(\mathbf{Wp} + b)$$



$$a = purelin(n) = purelin({}_1\mathbf{w}^T\mathbf{p} + b) = {}_1\mathbf{w}^T\mathbf{p} + b$$

$$a = {}_1\mathbf{w}^T\mathbf{p} + b = w_{1,1}p_1 + w_{1,2}p_2 + b$$

Mean Square Error

Training Set:

$$\{\mathbf{p}_1, \mathbf{t}_1\}, \{\mathbf{p}_2, \mathbf{t}_2\}, \dots, \{\mathbf{p}_Q, \mathbf{t}_Q\}$$

Input: $\mathbf{p}_q$ Target: $\mathbf{t}_q$

Notation:

$$\mathbf{x} = \begin{bmatrix} {}_1\mathbf{w} \\ b \end{bmatrix} \qquad \mathbf{z} = \begin{bmatrix} \mathbf{p} \\ 1 \end{bmatrix} \qquad a = {}_1\mathbf{w}^T\mathbf{p} + b \quad \Rightarrow \quad a = \mathbf{x}^T\mathbf{z}$$

Mean Square Error:

$$F(\mathbf{x}) = E[e^2] = E[(t - a)^2] = E[(t - \mathbf{x}^T\mathbf{z})^2]$$

Error Analysis

$$F(\mathbf{x}) = E[e^2] = E[(t-a)^2] = E[(t-\mathbf{x}^T\mathbf{z})^2]$$

$$F(\mathbf{x}) = E[t^2 - 2t\mathbf{x}^T\mathbf{z} + \mathbf{x}^T\mathbf{z}\mathbf{z}^T\mathbf{x}]$$

$$F(\mathbf{x}) = E[t^2] - 2\mathbf{x}^T\mathrm{E}[t\mathbf{z}] + \mathbf{x}^T E[\mathbf{z}\mathbf{z}^T]\mathbf{x}$$

$$\boxed{F(\mathbf{x}) = c - 2\mathbf{x}^T\mathbf{h} + \mathbf{x}^T\mathbf{R}\mathbf{x}}$$

$$c = E[t^2] \qquad \mathbf{h} = E[t\mathbf{z}] \qquad \mathbf{R} = E[\mathbf{z}\mathbf{z}^T]$$

The mean square error for the ADALINE Network is a quadratic function:

$$F(\mathbf{x}) = c + \mathbf{d}^T\mathbf{x} + \frac{1}{2}\mathbf{x}^T\mathbf{A}\mathbf{x}$$

$$\mathbf{d} = -2\mathbf{h} \qquad \mathbf{A} = 2\mathbf{R}$$

Stationary Point

Hessian Matrix:

$$\mathbf{A} = 2\mathbf{R}$$

The correlation matrix **R** must be at least positive semidefinite. If there are any zero eigenvalues, the performance index will either have a weak minumum or else no stationary point, otherwise there will be a unique global minimum **x***.

$$\nabla F(\mathbf{x}) = \nabla\left(c + \mathbf{d}^T\mathbf{x} + \frac{1}{2}\mathbf{x}^T\mathbf{A}\mathbf{x}\right) = \mathbf{d} + \mathbf{A}\mathbf{x} = -2\mathbf{h} + 2\mathbf{R}\mathbf{x}$$

$$-2\mathbf{h} + 2\mathbf{R}\mathbf{x} = \mathbf{0}$$

If R is positive definite:

$$\mathbf{x}^* = \mathbf{R}^{-1}\mathbf{h}$$

Approximate Steepest Descent

Approximate mean square error (one sample):

$$\hat{F}(\mathbf{x}) = (t(k) - a(k))^2 = e^2(k)$$

Approximate (stochastic) gradient:

$$\hat{\nabla} F(\mathbf{x}) = \nabla e^2(k)$$

$$[\nabla e^2(k)]_j = \frac{\partial e^2(k)}{\partial w_{1,j}} = 2e(k)\frac{\partial e(k)}{\partial w_{1,j}} \qquad j = 1, 2, \ldots, R$$

$$[\nabla e^2(k)]_{R+1} = \frac{\partial e^2(k)}{\partial b} = 2e(k)\frac{\partial e(k)}{\partial b}$$

Approximate Gradient Calculation

$$\frac{\partial e(k)}{\partial w_{1,j}} = \frac{\partial[t(k) - a(k)]}{\partial w_{1,j}} = \frac{\partial}{\partial w_{1,j}}[t(k) - ({}_1\mathbf{w}^T\mathbf{p}(k) + b)]$$

$$\frac{\partial e(k)}{\partial w_{1,j}} = \frac{\partial}{\partial w_{1,j}}\left[t(k) - \left(\sum_{i=1}^{R} w_{1,i}p_i(k) + b\right)\right]$$

$$\frac{\partial e(k)}{\partial w_{1,j}} = -p_j(k) \qquad \frac{\partial e(k)}{\partial b} = -1$$

$$\hat{\nabla}F(\mathbf{x}) = \nabla e^2(k) = -2e(k)\mathbf{z}(k)$$

LMS Algorithm

$$\mathbf{x}_{k+1} = \mathbf{x}_k - \alpha \nabla F(\mathbf{x}) \Big|_{\mathbf{x} = \mathbf{x}_k}$$

$$\mathbf{x}_{k+1} = \mathbf{x}_k + 2\alpha e(k) \mathbf{z}(k)$$

$${}_1\mathbf{w}(k+1) = {}_1\mathbf{w}(k) + 2\alpha e(k) \mathbf{p}(k)$$

$$b(k+1) = b(k) + 2\alpha e(k)$$

Multiple-Neuron Case

$$ {}_i\mathbf{w}(k+1) = {}_i\mathbf{w}(k) + 2\alpha e_i(k)\mathbf{p}(k) $$

$$ b_i(k+1) = b_i(k) + 2\alpha e_i(k) $$

Matrix Form:

$$ \mathbf{W}(k+1) = \mathbf{W}(k) + 2\alpha\mathbf{e}(k)\mathbf{p}^T(k) $$

$$ \mathbf{b}(k+1) = \mathbf{b}(k) + 2\alpha\mathbf{e}(k) $$

Analysis of Convergence

$$\mathbf{x}_{k+1} = \mathbf{x}_k + 2\alpha e(k)\mathbf{z}(k)$$

$$E[\mathbf{x}_{k+1}] = E[\mathbf{x}_k] + 2\alpha E[e(k)\mathbf{z}(k)]$$

$$E[\mathbf{x}_{k+1}] = E[\mathbf{x}_k] + 2\alpha\{E[t(k)\mathbf{z}(k)] - E[(\mathbf{x}_k^T\mathbf{z}(k))\mathbf{z}(k)]\}$$

$$E[\mathbf{x}_{k+1}] = E[\mathbf{x}_k] + 2\alpha\{E[t_k\mathbf{z}(k)] - E[(\mathbf{z}(k)\mathbf{z}^T(k))\mathbf{x}_k]\}$$

$$E[\mathbf{x}_{k+1}] = E[\mathbf{x}_k] + 2\alpha\{\mathbf{h} - \mathbf{R}E[\mathbf{x}_k]\}$$

$$E[\mathbf{x}_{k+1}] = \underbrace{[\mathbf{I} - 2\alpha\mathbf{R}]}_{\text{}}E[\mathbf{x}_k] + 2\alpha\mathbf{h}$$

For stability, the eigenvalues of this matrix must fall inside the unit circle.

Conditions for Stability

$$\left|eig([\mathbf{I} - 2\alpha\mathbf{R}])\right| = \left|1 - 2\alpha\lambda_i\right| < 1$$

(where λ_i is an eigenvalue of **R**)

Since $\lambda_i > 0$, $1 - 2\alpha\lambda_i < 1$.

Therefore the stability condition simplifies to

$$1 - 2\alpha\lambda_i > -1$$

$$\alpha < 1/\lambda_i \quad \text{for all } i$$

$$\boxed{0 < \alpha < 1/\lambda_{max}}$$

Steady State Response

$$E[\mathbf{x}_{k+1}] = [\mathbf{I} - 2\alpha\mathbf{R}]E[\mathbf{x}_k] + 2\alpha\mathbf{h}$$

If the system is stable, then a steady state condition will be reached.

$$E[\mathbf{x}_{ss}] = [\mathbf{I} - 2\alpha\mathbf{R}]E[\mathbf{x}_{ss}] + 2\alpha\mathbf{h}$$

The solution to this equation is

$$E[\mathbf{x}_{ss}] = \mathbf{R}^{-1}\mathbf{h} = \mathbf{x}^*$$

This is also the strong minimum of the performance index.

Example

$$\text{Banana} \quad \left\{ \mathbf{p}_1 = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}, \mathbf{t}_1 = \begin{bmatrix} -1 \end{bmatrix} \right\} \qquad \text{Apple} \quad \left\{ \mathbf{p}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \mathbf{t}_2 = \begin{bmatrix} 1 \end{bmatrix} \right\}$$

$$\mathbf{R} = E[\mathbf{p}\mathbf{p}^T] = \frac{1}{2}\mathbf{p}_1\mathbf{p}_1^T + \frac{1}{2}\mathbf{p}_2\mathbf{p}_2^T$$

$$\mathbf{R} = \frac{1}{2}\begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}\begin{bmatrix} -1 & 1 & -1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}\begin{bmatrix} 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\lambda_1 = 1.0, \qquad \lambda_2 = 0.0, \qquad \lambda_3 = 2.0$$

$$\alpha < \frac{1}{\lambda_{max}} = \frac{1}{2.0} = 0.5$$

Iteration One

Banana $$a(0) = \mathbf{W}(0)\mathbf{p}(0) = \mathbf{W}(0)\mathbf{p}_1 = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} = 0$$

$$e(0) = t(0) - a(0) = t_1 - a(0) = -1 - 0 = -1$$

$$\mathbf{W}(1) = \mathbf{W}(0) + 2\alpha e(0)\mathbf{p}^T(0)$$

$$\mathbf{W}(1) = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} + 2(0.2)(-1) \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}^T = \begin{bmatrix} 0.4 & -0.4 & 0.4 \end{bmatrix}$$

Iteration Two

Apple $$a(1) = \mathbf{W}(1)\mathbf{p}(1) = \mathbf{W}(1)\mathbf{p}_2 = \begin{bmatrix} 0.4 & -0.4 & 0.4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = -0.4$$

$$e(1) = t(1) - a(1) = t_2 - a(1) = 1 - (-0.4) = 1.4$$

$$\mathbf{W}(2) = \begin{bmatrix} 0.4 & -0.4 & 0.4 \end{bmatrix} + 2(0.2)(1.4) \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}^T = \begin{bmatrix} 0.96 & 0.16 & -0.16 \end{bmatrix}$$

Iteration Three

$$a(2) = \mathbf{W}(2)\mathbf{p}(2) = \mathbf{W}(2)\mathbf{p}_1 = \begin{bmatrix} 0.96 & 0.16 & -0.16 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} = -0.64$$

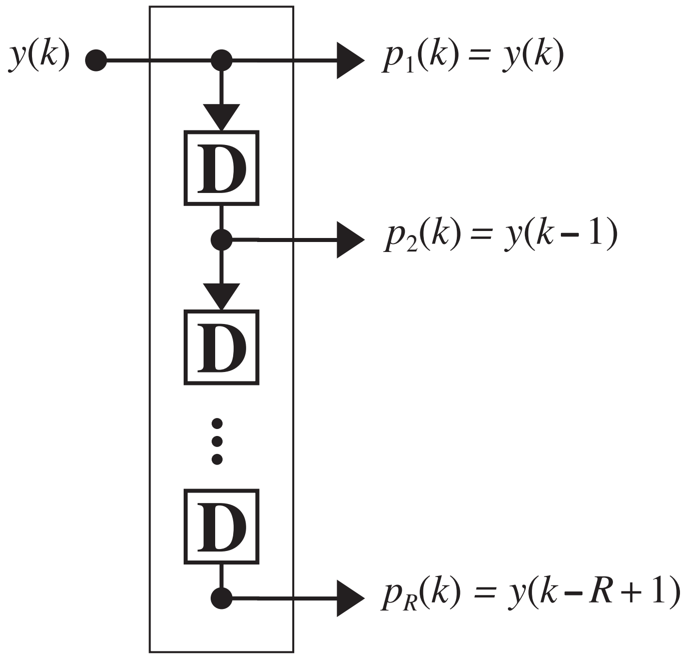
$$e(2) = t(2) - a(2) = t_1 - a(2) = -1 - (-0.64) = -0.36$$

$$\mathbf{W}(3) = \mathbf{W}(2) + 2\alpha e(2)\mathbf{p}^T(2) = \begin{bmatrix} 1.1040 & 0.0160 & -0.0160 \end{bmatrix}$$

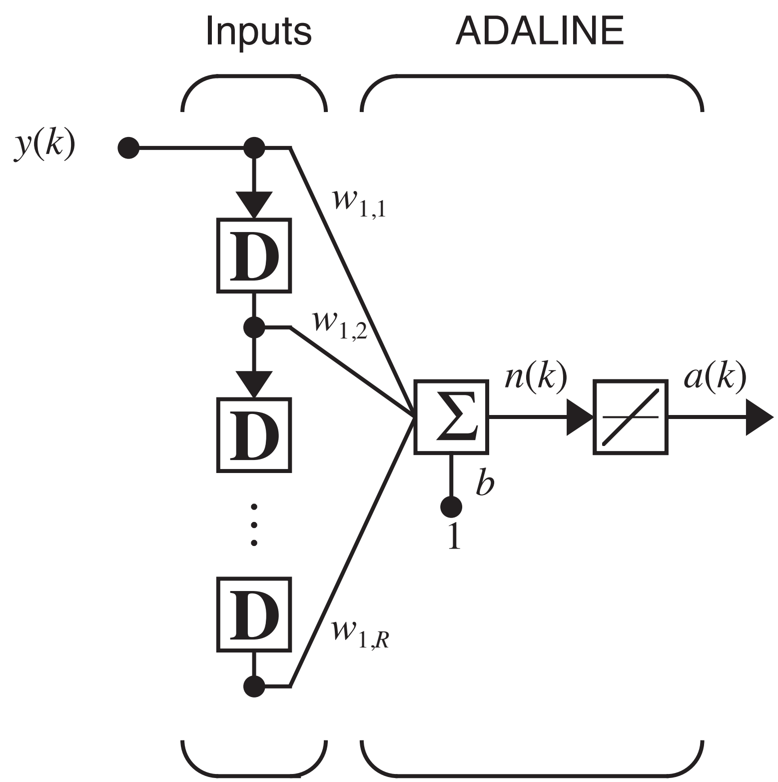
$$\mathbf{W}(\infty) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

Adaptive Filtering

Tapped Delay Line



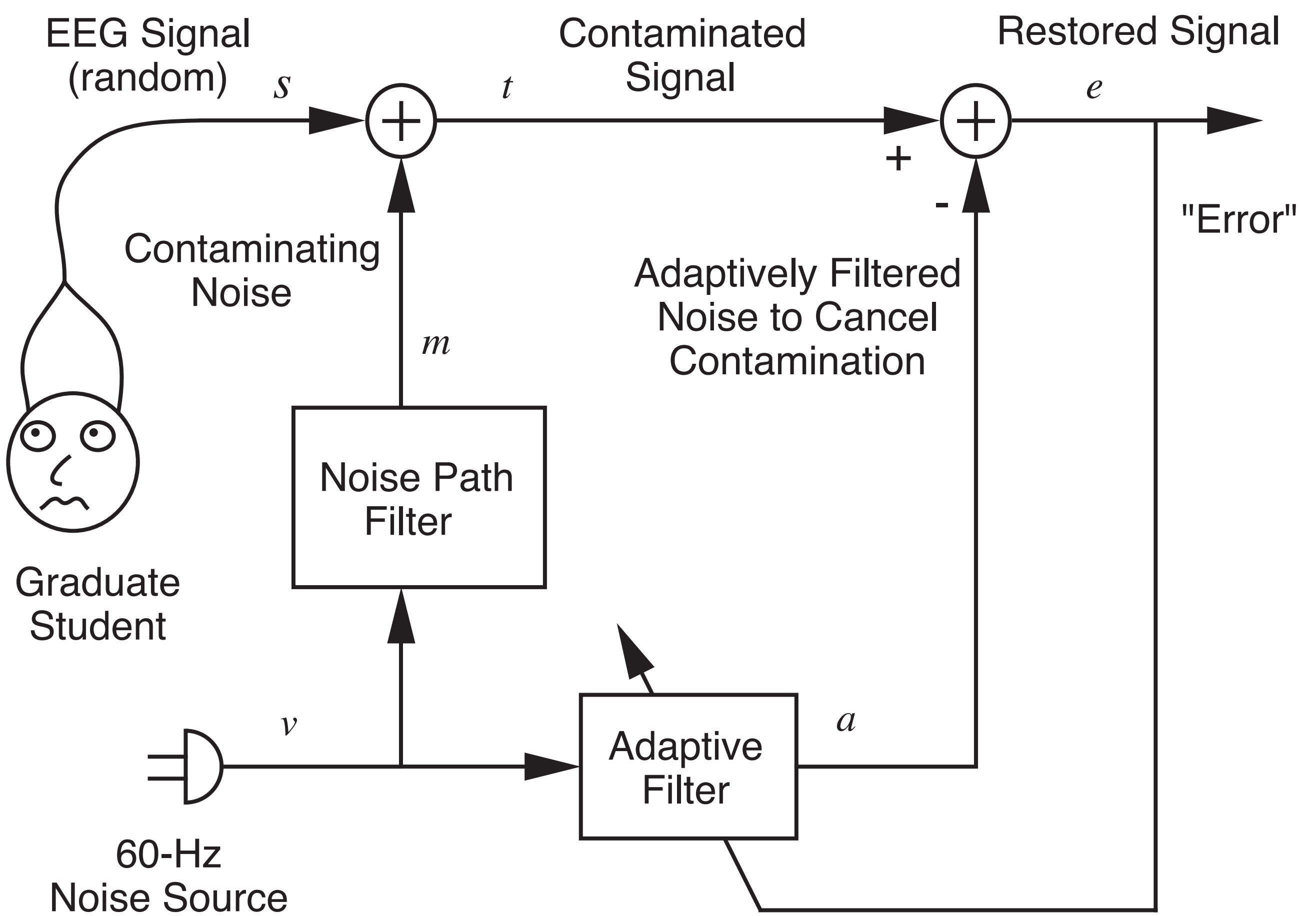
Adaptive Filter



$$a(k) = purelin(\mathbf{W}\mathbf{p}(k) + b)$$

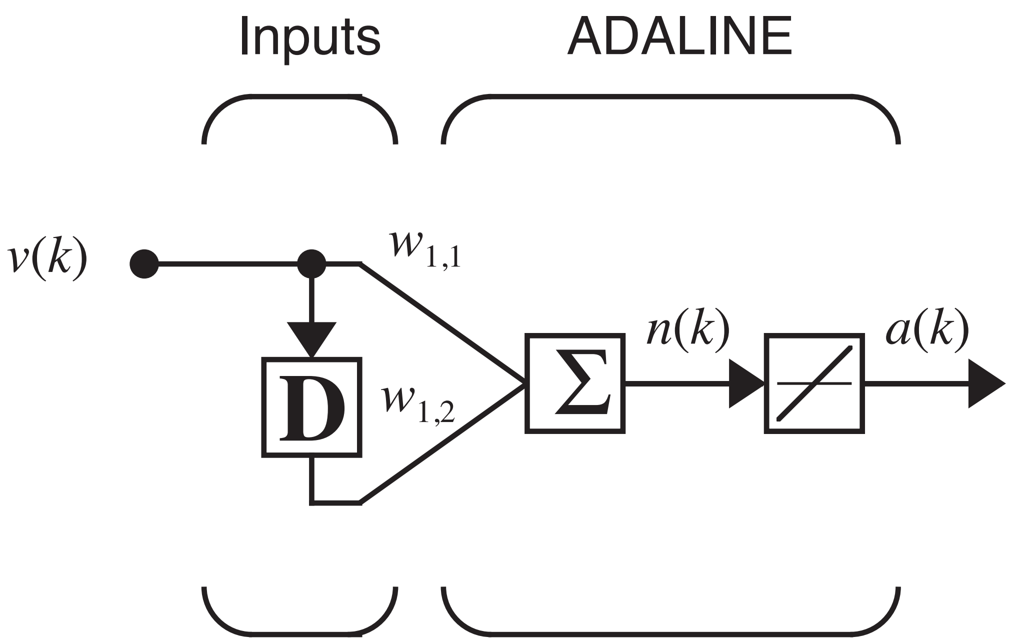
$$a(k) = purelin(\mathbf{W}\mathbf{p} + b) = \sum_{i=1}^{R} w_{1,i}\, y(k - i + 1) + b$$

Example: Noise Cancellation



Adaptive Filter Adjusts to Minimize Error (and in doing this removes 60-Hz noise from contaminated signal)

Noise Cancellation Adaptive Filter



$$a(k) = w_{1,1}\,v(k) + w_{1,2}\,v(k-1)$$

Correlation Matrix

$$\mathbf{R} = [\mathbf{z}\mathbf{z}^T] \qquad \mathbf{h} = E[t\mathbf{z}]$$

$$\mathbf{z}(k) = \begin{bmatrix} v(k) \\ v(k-1) \end{bmatrix}$$

$$t(k) = s(k) + m(k)$$

$$\mathbf{R} = \begin{bmatrix} E[v^2(k)] & E[v(k)v(k-1)] \\ E[v(k-1)v(k)] & E[v^2(k-1)] \end{bmatrix}$$

$$\mathbf{h} = \begin{bmatrix} E[(s(k) + m(k))v(k)] \\ E[(s(k) + m(k))v(k-1)] \end{bmatrix}$$

Signals

$$v(k) = 1.2 \sin\left(\frac{2\pi k}{3}\right) \qquad m(k) = 1.2 \sin\left(\frac{2\pi k}{3} - \frac{3\pi}{4}\right)$$

$$E[v^2(k)] = (1.2)^2 \frac{1}{3} \sum_{k=1}^{3} \left(\sin\left(\frac{2\pi k}{3}\right)\right)^2 = (1.2)^2 0.5 = 0.72$$

$$E[v^2(k-1)] = E[v^2(k)] = 0.72$$

$$E[v(k)v(k-1)] = \frac{1}{3} \sum_{k=1}^{3} \left(1.2 \sin\frac{2\pi k}{3}\right)\left(1.2 \sin\frac{2\pi(k-1)}{3}\right)$$

$$= (1.2)^2 0.5 \cos\left(\frac{2\pi}{3}\right) = -0.36$$

$$\mathbf{R} = \begin{bmatrix} 0.72 & -0.36 \\ -0.36 & 0.72 \end{bmatrix}$$

Stationary Point

$$E[(s(k)+m(k))v(k)] = E[s(k)v(k)] + E[m(k)v(k)]$$

$$\Downarrow$$

$$0$$

$$E[m(k)v(k)] = \frac{1}{3}\sum_{k=1}^{3}\left(1.2\ \sin\left(\frac{2\pi k}{3}-\frac{3\pi}{4}\right)\right)\left(1.2\sin\frac{2\pi k}{3}\right) = -0.51$$

$$E[(s(k)+m(k))v(k-1)] = E[s(k)v(k-1)] + E[m(k)v(k-1)]$$

$$\Downarrow$$

$$0$$

$$E[m(k)v(k-1)] = \frac{1}{3}\sum_{k=1}^{3}\left(1.2\ \sin\left(\frac{2\pi k}{3}-\frac{3\pi}{4}\right)\right)\left(1.2\ \sin\frac{2\pi(k-1)}{3}\right) = 0.70$$

$$\mathbf{h} = \begin{bmatrix} E[(s(k)+m(k))v(k)] \\ E[(s(k)+m(k))v(k-1)] \end{bmatrix} \Rightarrow \mathbf{h} = \begin{bmatrix} -0.51 \\ 0.70 \end{bmatrix}$$

$$\mathbf{x}^* = \mathbf{R}^{-1}\mathbf{h} = \begin{bmatrix} 0.72 & -0.36 \\ -0.36 & 0.72 \end{bmatrix}^{-1}\begin{bmatrix} -0.51 \\ 0.70 \end{bmatrix} = \begin{bmatrix} -0.30 \\ 0.82 \end{bmatrix}$$

Performance Index

$$F(\mathbf{x}) = c - 2\mathbf{x}^T\mathbf{h} + \mathbf{x}^T\mathbf{R}\mathbf{x}$$

$$c = E[t^2(k)] = E[(s(k) + m(k))^2]$$

$$c = E[s^2(k)] + 2E[s(k)m(k)] + E[m^2(k)]$$

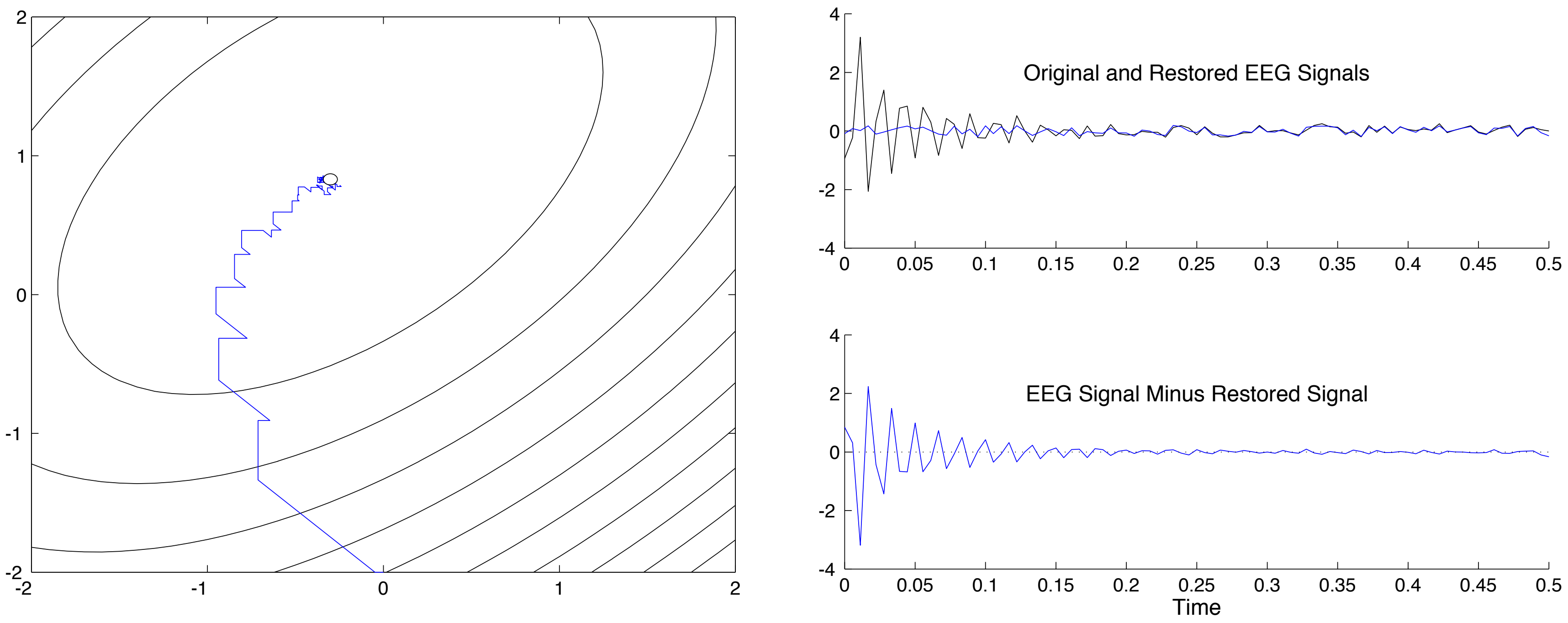
$$E[s^2(k)] = \frac{1}{0.4}\int_{-0.2}^{0.2} s^2\,ds = \frac{1}{3(0.4)}s^3\Big|_{-0.2}^{0.2} = 0.0133$$

$$E[m^2(k)] = \frac{1}{3}\sum_{k=1}^{3}\left\{1.2\,\sin\left(\frac{2\pi}{3} - \frac{3\pi}{4}\right)\right\}^2 = 0.72$$

$$c = 0.0133 + 0.72 = 0.7333$$

$$F(\mathbf{x}^*) = 0.7333 - 2(0.72) + 0.72 = 0.0133$$

LMS Response



Echo Cancellation