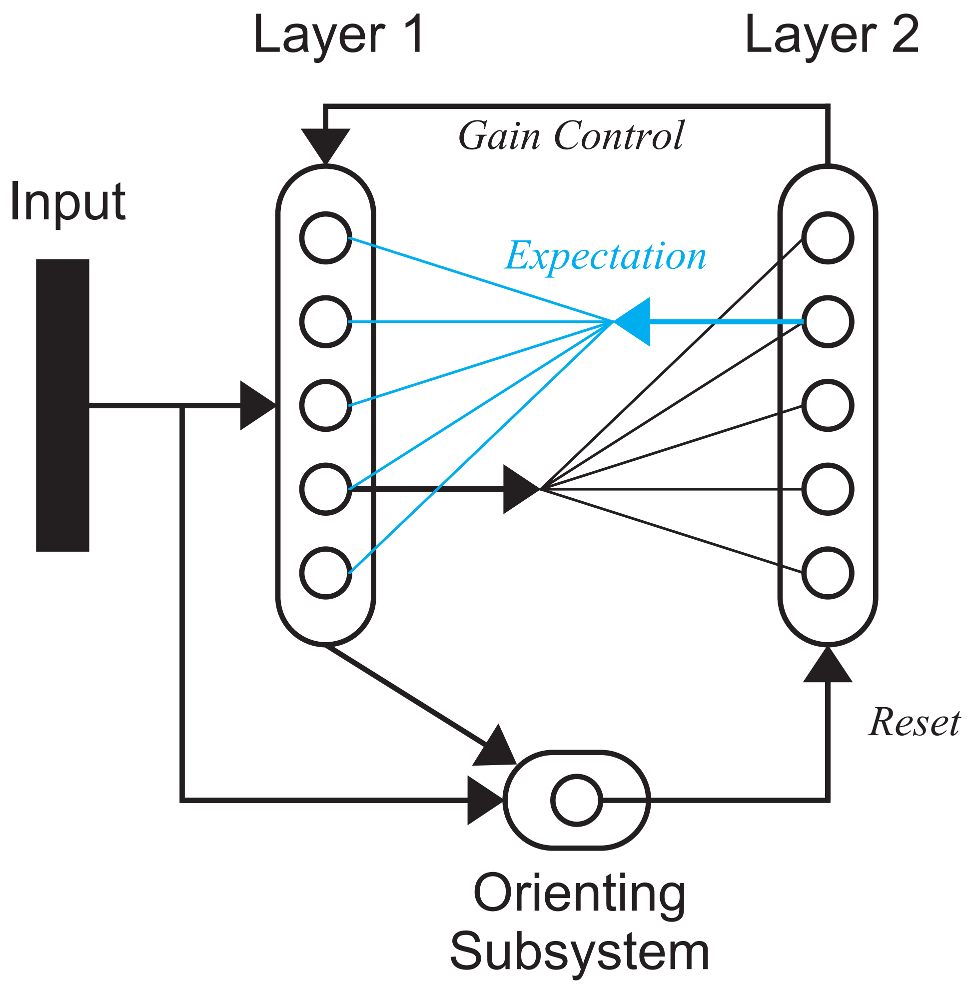
Adaptive Resonance Theory (ART)

Basic ART Architecture



ART Subsystems

Layer 1
- Normalization
- Comparison of input pattern and expectation

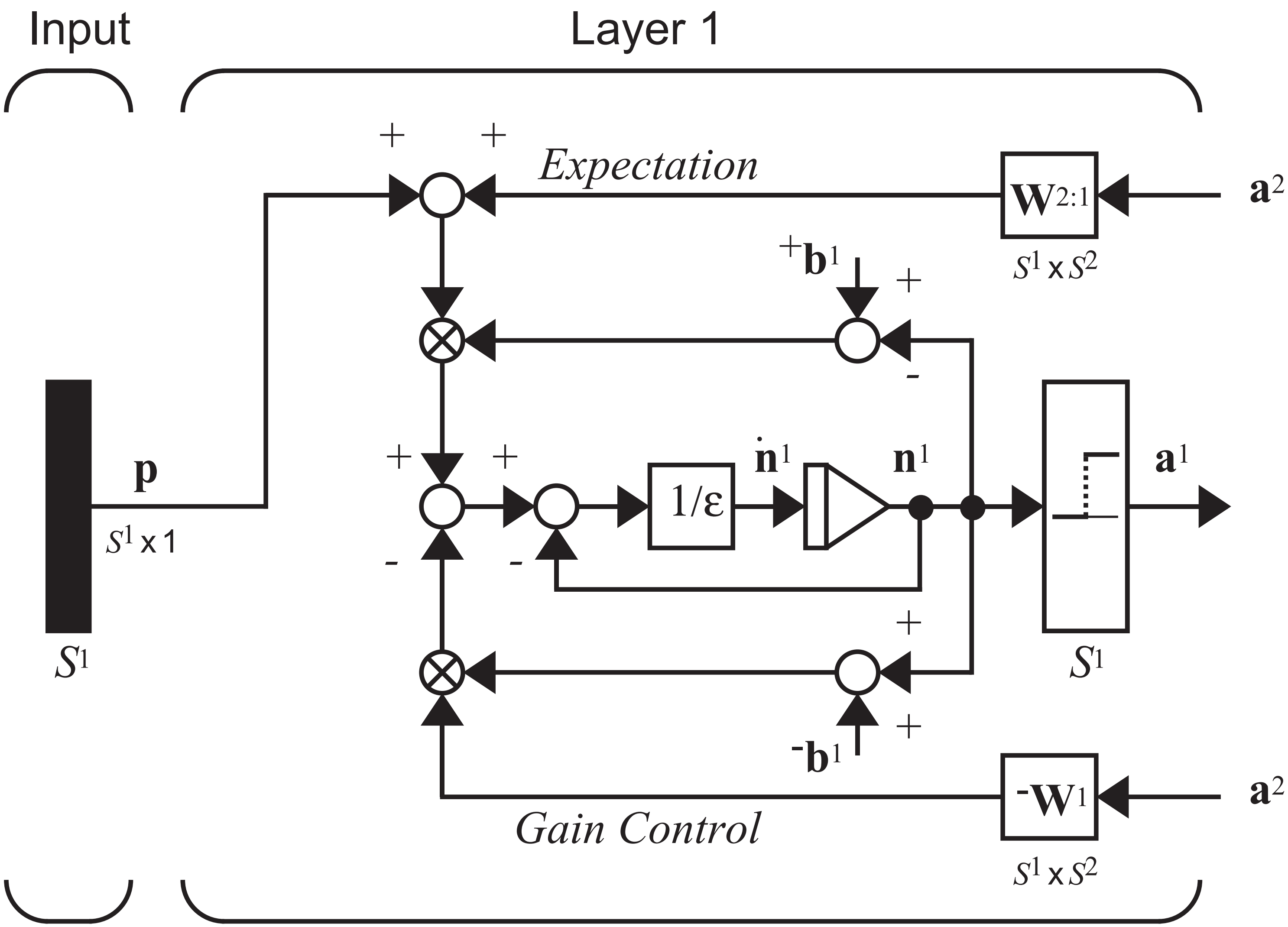
L1-L2 Connections (Instars)
- Perform clustering operation.
- Each row of $W^{1:2}$ is a prototype pattern.

Layer 2
- Competition, contrast enhancement

L2-L1 Connections (Outstars)
- Expectation
- Perform pattern recall.
- Each column of $W^{2:1}$ is a prototype pattern

Orienting Subsystem
- Causes a reset when expectation does not match input
- Disables current winning neuron

Layer 1



$$\varepsilon\, d\mathbf{n}^1/dt = -\mathbf{n}^1 + ({}^{+}\mathbf{b}^1 - \mathbf{n}^1)\{\mathbf{p} + \mathbf{W}^{2:1}\mathbf{a}^2\} - (\mathbf{n}^1 + {}^{-}\mathbf{b}^1)[{}^{-}\mathbf{W}^1]\mathbf{a}^2$$

Layer 1 Operation

Shunting Model

$$\varepsilon \frac{d\mathbf{n}^1(t)}{dt} = -\mathbf{n}^1(t) + ({}^{+}\mathbf{b}^1 - \mathbf{n}^1(t))\{\underbrace{\mathbf{p} + \mathbf{W}^{2:1}\mathbf{a}^2(t)}\} - (\mathbf{n}^1(t) + {}^{-}\mathbf{b}^1)[\underbrace{{}^{-}\mathbf{W}^1]\mathbf{a}^2(t)}$$

Excitatory Input
(Comparison with Expectation)

Inhibitory Input
(Gain Control)

$$\mathbf{a}^1 = \mathbf{hardlim}^{+}(\mathbf{n}^1)$$

$$hardlim^{+}(n) = \begin{cases} 1, & n > 0 \\ 0, & n \le 0 \end{cases}$$

Excitatory Input to Layer 1

$$\mathbf{p} + \mathbf{W}^{2:1}\mathbf{a}^{2}(t)$$

Suppose that neuron *j* in Layer 2 has won the competition:

$$\mathbf{W}^{2:1}\mathbf{a}^{2} = \begin{bmatrix} \mathbf{w}_1^{2:1} & \mathbf{w}_2^{2:1} & \dots & \mathbf{w}_j^{2:1} & \dots & \mathbf{w}_{S^2}^{2:1} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \end{bmatrix} = \mathbf{w}_j^{2:1}$$

(*j*th column of $\mathbf{W}^{2:1}$)

Therefore the excitatory input is the sum of the input pattern and the L2-L1 expectation:

$$\mathbf{p} + \mathbf{W}^{2:1}\mathbf{a}^{2} = \mathbf{p} + \mathbf{w}_j^{2:1}$$

Inhibitory Input to Layer 1

Gain Control

$$[{}^{-}\mathbf{W}^1]\mathbf{a}^2(t)$$

$$
{}^{-}\mathbf{W}^1 = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}
$$

The gain control will be one when Layer 2 is active (one neuron has won the competition), and zero when Layer 2 is inactive (all neurons having zero output).

19 Steady State Analysis: Case I

$$\varepsilon \frac{dn_i^1}{dt} = -n_i^1 + ({}^{+}b^1 - n_i^1)\left\{p_i + \sum_{j=1}^{S^2} w_{i,j}^{2:1} a_j^2\right\} - (n_i^1 + {}^{-}b^1)\sum_{j=1}^{S^2} a_j^2$$

Case I: Layer 2 inactive (each $a^2_j = 0$)

$$\varepsilon \frac{dn_i^1}{dt} = -n_i^1 + ({}^{+}b^1 - n_i^1)\{p_{\mathrm{i}}\}$$

In steady state:

$$0 = -n_i^1 + ({}^{+}b^1 - n_i^1)p_i = -(1 + p_i)n_i^1 + {}^{+}b^1 p_i \Rightarrow n_i^1 = \frac{{}^{+}b^1 p_i}{1 + p_i}$$

Therefore, if Layer 2 is inactive:

$$\mathbf{a}^1 = \mathbf{p}$$

19 Steady State Analysis: Case II

Case II: Layer 2 active (one $a^2_j = 1$)

$$\varepsilon \frac{dn_i^1}{dt} = -n_i^1 + ({}^{+}b^1 - n_i^1)\{p_i + w_{i,j}^{2:1}\} - (n_i^1 + {}^{-}b^1)$$

In steady state:

$$\begin{aligned} 0 &= -n_i^1 + ({}^{+}b^1 - n_i^1)\{p_i + w_{i,j}^{2:1}\} - (n_i^1 + {}^{-}b^1) \\ &= -(1 + p_i + w_{i,j}^{2:1} + 1)n_i^1 + ({}^{+}b^1(p_i + w_{i,j}^{2:1}) - {}^{-}b^1) \end{aligned} \quad \Longrightarrow \quad n_i^1 = \frac{{}^{+}b^1(p_i + w_{i,j}^{2:1}) - {}^{-}b^1}{2 + p_i + w_{i,j}^{2:1}}$$

We want Layer 1 to combine the input vector with the expectation from Layer 2, using a logical AND operation:

- $n^1_i < 0$, if either $w^{2:1}_{i,j}$ or p_i is equal to zero.
- $n^1_i > 0$, if both $w^{2:1}_{i,j}$ or p_i are equal to one.

$$\left.\begin{aligned} {}^{+}b^1(2) - {}^{-}b^1 &> 0 \\ {}^{+}b^1 - {}^{-}b^1 &< 0 \end{aligned}\right\} \quad {}^{+}b^1(2) > {}^{-}b^1 > {}^{+}b^1$$

Therefore, if Layer 2 is active, and the biases satisfy these conditions:

$$\mathbf{a}^1 = \mathbf{p} \cap \mathbf{w}_j^{2:1}$$

19

Layer 1 Summary

If Layer 2 is inactive (each $a^2{}_j = 0$)

$$\mathbf{a}^1 = \mathbf{p}$$

If Layer 2 is active (one $a^2{}_j = 1$)

$$\mathbf{a}^1 = \mathbf{p} \cap \mathbf{w}_j^{2:1}$$

Layer 1 Example

$\varepsilon = 1,\ {}^{+}b^1 = 1$ and ${}^{-}b^1 = 1.5$ $\qquad \mathbf{W}^{2:1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad \mathbf{p} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Assume that Layer 2 is active, and neuron 2 won the competition.

$$(0.1)\frac{dn_1^1}{dt} = -n_1^1 + (1 - n_1^1)\{p_1 + w_{1,2}^{2:1}\} - (n_1^1 + 1.5)$$

$$= -n_1^1 + (1 - n_1^1)\{0 + 1\} - (n_1^1 + 1.5) = -3n_1^1 - 0.5$$

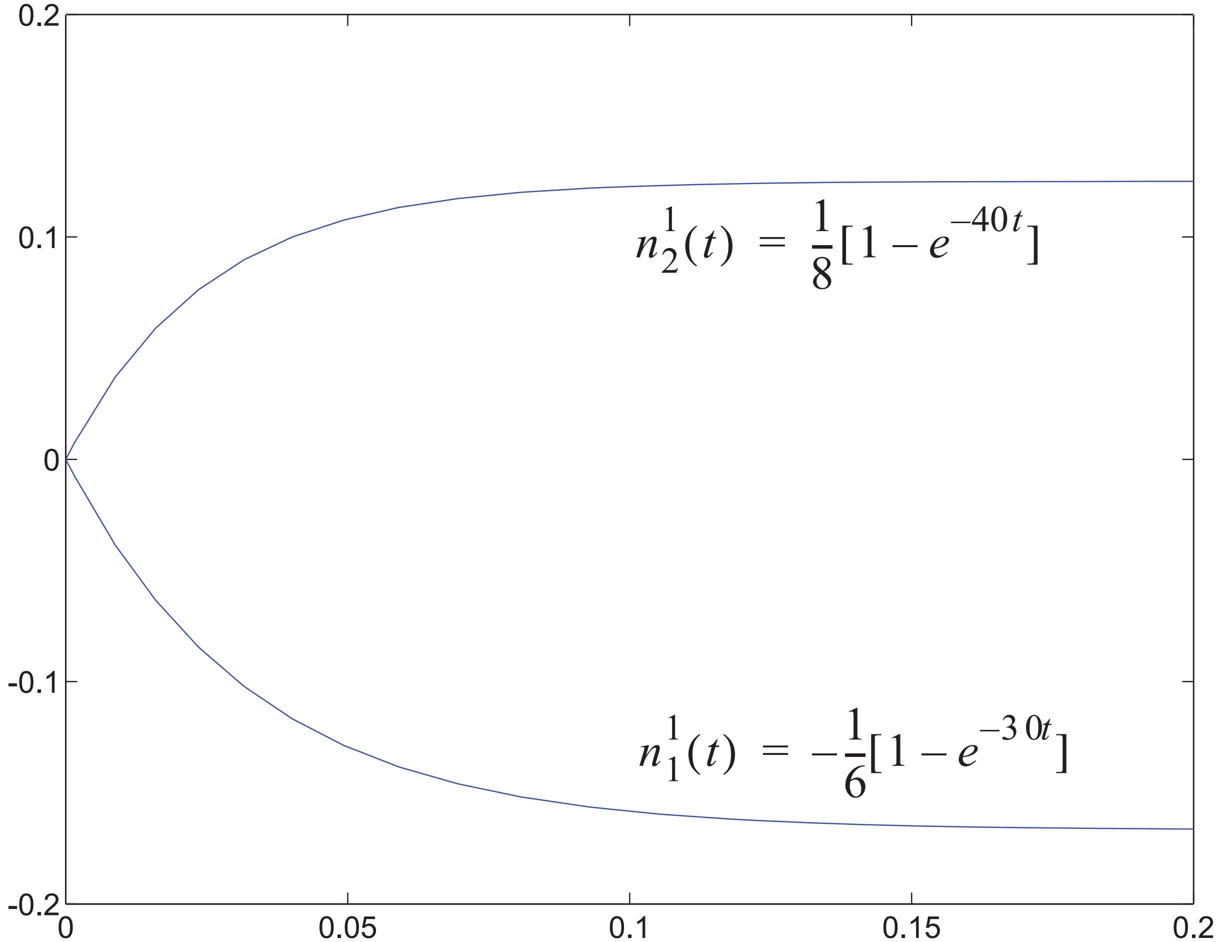
$$\frac{dn_1^1}{dt} = -30n_1^1 - 5$$

$$(0.1)\frac{dn_2^1}{dt} = -n_2^1 + (1 - n_2^1)\{p_2 + w_{2,2}^{2:1}\} - (n_2^1 + 1.5)$$

$$= -n_2^1 + (1 - n_2^1)\{1 + 1\} - (n_2^1 + 1.5) = -4n_2^1 + 0.5$$

$$\frac{dn_2^1}{dt} = -40n_2^1 + 5$$

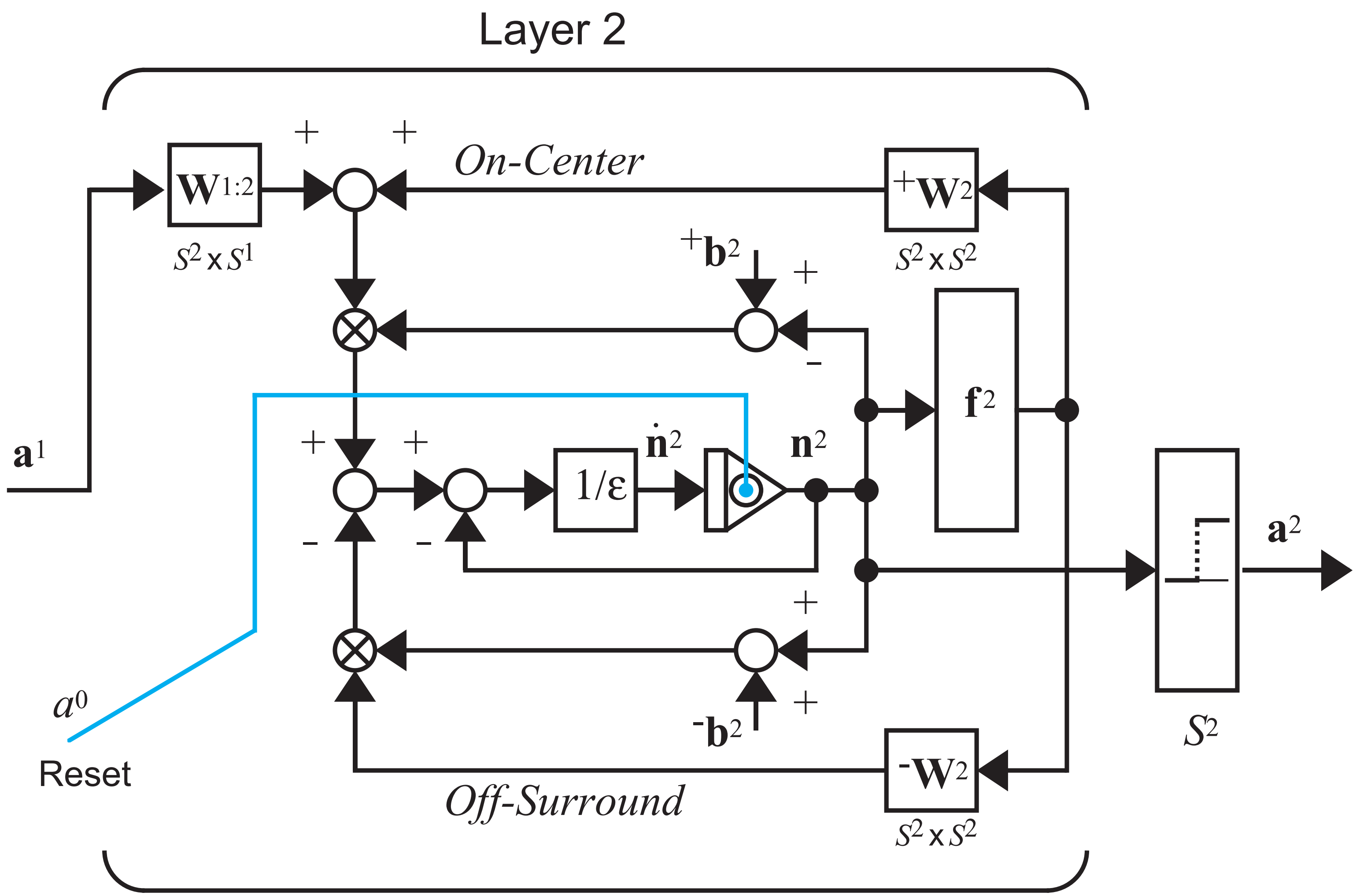
Example Response



$$\mathbf{p} \cap \mathbf{w}_2^{2:1} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cap \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \mathbf{a}^1$$

19

Layer 2



$$\varepsilon \, d\mathbf{n}^2/dt = -\mathbf{n}^2 + ({}^{+}\mathbf{b}^2 - \mathbf{n}^2)\{[{}^{+}\mathbf{W}^2]\mathbf{f}^2(\mathbf{n}^2) + \mathbf{W}^{1:2}\mathbf{a}^1\}$$
$$- (\mathbf{n}^2 + {}^{-}\mathbf{b}^2)[{}^{-}\mathbf{W}^2]\mathbf{f}^2(\mathbf{n}^2)$$

Layer 2 Operation

Shunting Model

$$\varepsilon \frac{d\mathbf{n}^2(t)}{dt} = -\mathbf{n}^2(t)$$

$$+ ({}^{+}\mathbf{b}^2 - \mathbf{n}^2(t))\underbrace{\{\overbrace{[{}^{+}\mathbf{W}^2]\mathbf{f}^2(\mathbf{n}^2(t))}^{\text{On-Center Feedback}} + \overbrace{\mathbf{W}^{1:2}\mathbf{a}^1}^{\text{Adaptive Instars}}\}}_{\text{Excitatory Input}}$$

$$- (\mathbf{n}^2(\mathrm{t}) + {}^{-}\mathbf{b}^2)\underbrace{\overbrace{[{}^{-}\mathbf{W}^2]\mathbf{f}^2(\mathbf{n}^2(t))}^{\text{Off-Surround Feedback}}}_{\text{Inhibitory Input}}$$

Layer 2 Example

$$\varepsilon = 0.1 \qquad {}^{+}\mathbf{b}^2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad {}^{-}\mathbf{b}^2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \mathbf{W}^{1:2} = \begin{bmatrix} ({}_1\mathbf{w}^{1:2})^T \\ ({}_2\mathbf{w}^{1:2})^T \end{bmatrix} = \begin{bmatrix} 0.5 & 0.5 \\ 1 & 0 \end{bmatrix}$$

$$f^2(n) = \begin{cases} 10(n)^2, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

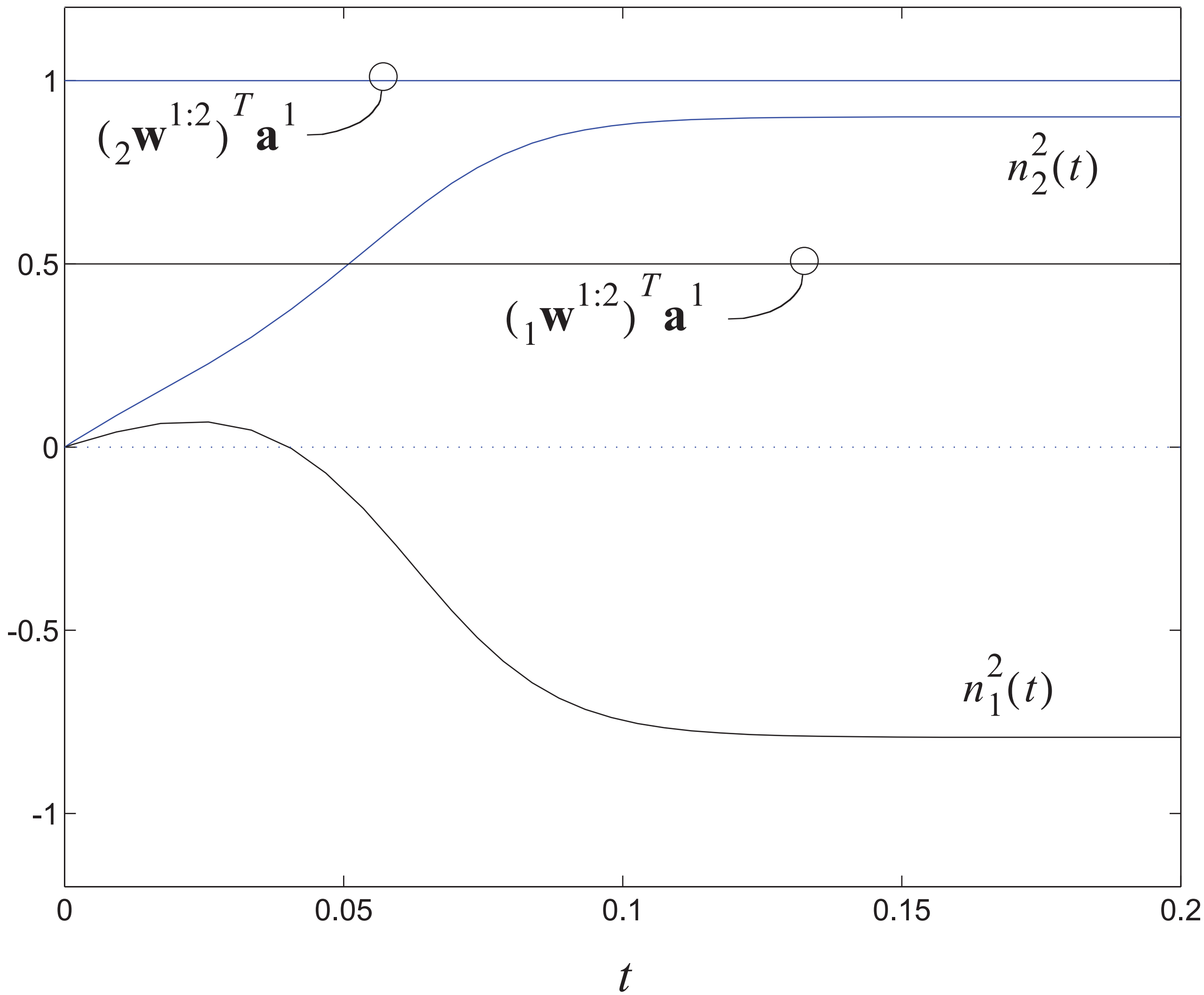
(Faster than linear, winner-take-all)

$$(0.1)\frac{dn_1^2(t)}{dt} = -n_1^2(t) + (1 - n_1^2(t))\left\{ f^2(n_1^2(t)) + ({}_1\mathbf{w}^{1:2})^T \mathbf{a}^1 \right\} - (n_1^2(t) + 1) f^2(n_2^2(t))$$

$$(0.1)\frac{\mathrm{d}n_2^2(t)}{\mathrm{d}t} = -n_2^2(t) + (1 - n_2^2(t))\left\{ f^2(n_2^2(t)) + ({}_2\mathbf{w}^{1:2})^T \mathbf{a}^1 \right\} - (n_2^2(t) + 1) f^2(n_1^2(t)) .$$

Example Response

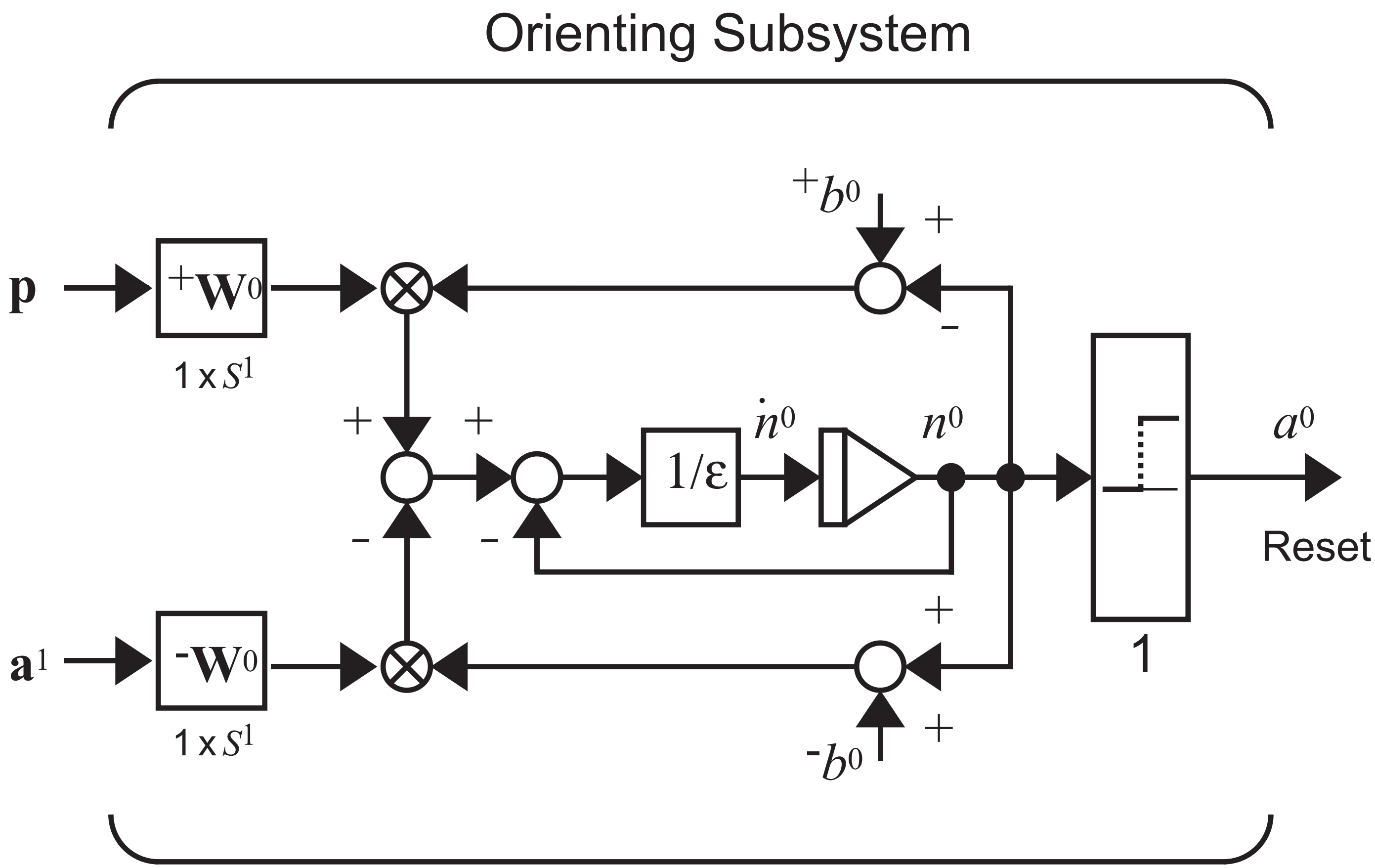
$$\mathbf{a}^1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \mathbf{a}^2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$



Layer 2 Summary

$$a_i^2 = \begin{cases} 1\,, & \text{if}\left(({}_i\mathbf{w}^{1:2})^T\mathbf{a}^1 = max[({}_j\mathbf{w}^{1:2})^T\mathbf{a}^1]\right) \\ 0\,, & \text{otherwise} \end{cases}$$

Orienting Subsystem



$$\varepsilon\, dn^0/dt = -n^0 + ({}^{+}b^0 - n^0)[{}^{+}\mathbf{W}^0]\mathbf{p} - (n^0 + {}^{-}b^0)[{}^{-}\mathbf{W}^0]\mathbf{a}^1$$

Purpose: Determine if there is a sufficient match between the L2-L1 expectation ($\mathbf{a}^1$) and the input pattern ($\mathbf{p}$).

Orienting Subsystem Operation

$$\varepsilon \frac{dn^0(t)}{dt} = -n^0(t) + ({}^{+}b^0 - n^0(t))\{{}^{+}\mathbf{W}^0\mathbf{p}\} - (n^0(t) + {}^{-}b^0)\{{}^{-}\mathbf{W}^0\mathbf{a}^1\}$$

Excitatory Input

$${}^{+}\mathbf{W}^0\mathbf{p} = \begin{bmatrix}\alpha & \alpha & \dots & \alpha\end{bmatrix}\mathbf{p} = \alpha \sum_{j=1}^{S^1} p_j = \alpha\|\mathbf{p}\|^2$$

Inhibitory Input

$${}^{-}\mathbf{W}^0\mathbf{a}^1 = \begin{bmatrix}\beta & \beta & \dots & \beta\end{bmatrix}\mathbf{a}^1 = \beta \sum_{j=1}^{S^1} a_j^1(t) = \beta\|\mathbf{a}^1\|^2$$

When the excitatory input is larger than the inhibitory input, the Orienting Subsystem will be driven on.

Steady State Operation

$$0 = -n^0 + ({}^{+}b^0 - n^0)\{\alpha\|\mathbf{p}\|^2\} - (n^0 + {}^{-}b^0)\left\{\beta\left\|\mathbf{a}^1\right\|^2\right\}$$

$$= -(1 + \alpha\|\mathbf{p}\|^2 + \beta\left\|\mathbf{a}^1\right\|^2)n^0 + {}^{+}b^0(\alpha\|\mathbf{p}\|^2) - {}^{-}b^0(\beta\left\|\mathbf{a}^1\right\|^2)$$

$$n^0 = \frac{{}^{+}b^0(\alpha\|\mathbf{p}\|^2) - {}^{-}b^0(\beta\left\|\mathbf{a}^1\right\|^2)}{(1 + \alpha\|\mathbf{p}\|^2 + \beta\left\|\mathbf{a}^1\right\|^2)}$$

Let $\;{}^{+}b^0 = {}^{-}b^0 = 1$

$$n^0 > 0 \quad \text{if} \quad \frac{\left\|\mathbf{a}^1\right\|^2}{\|\mathbf{p}\|^2} < \frac{\alpha}{\beta} = \rho$$

RESET

ρ → Vigilance

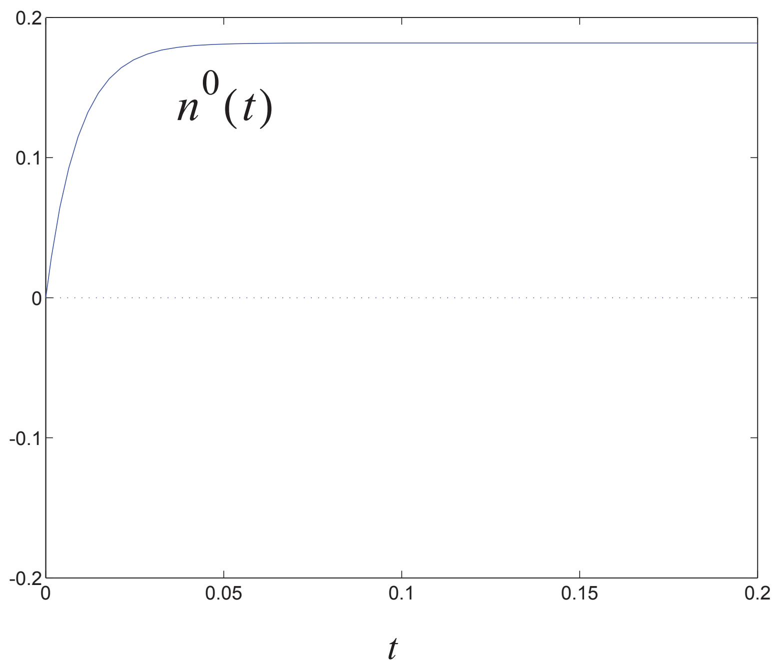
Since $\mathbf{a}^1 = \mathbf{p} \cap \mathbf{w}_j^{2:1}$, a reset will occur when there is enough of a mismatch between $\mathbf{p}$ and $\mathbf{w}_j^{2:1}$.

Orienting Subsystem Example

$$\varepsilon = 0.1,\ \alpha = 3,\ \beta = 4\ (\rho = 0.75) \qquad \mathbf{p} = \begin{bmatrix}1\\1\end{bmatrix} \qquad \mathbf{a}^1 = \begin{bmatrix}1\\0\end{bmatrix}$$

$$(0.1)\frac{dn^0(t)}{dt} = -n^0(t) + (1 - n^0(t))\{3(p_1 + p_2)\} - (n^0(t) + 1)\{4(a_1^1 + a_2^1)\}$$

$$\frac{dn^0(t)}{dt} = -110n^0(t) + 20$$

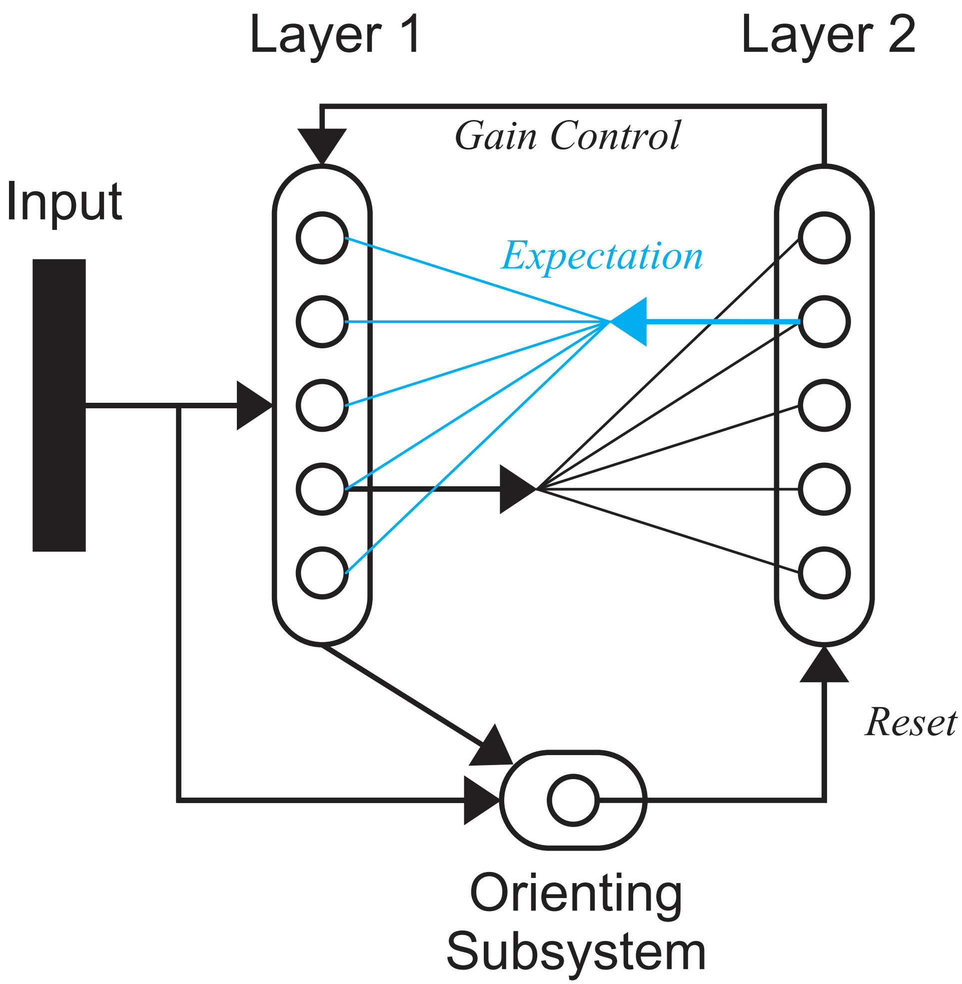


19

Orienting Subsystem Summary

$$a^0 = \begin{cases} 1 \,, & \text{if}\left[\|\mathbf{a}^1\|^2 / \|\mathbf{p}\|^2 < \rho \right] \\ 0 \,, & \text{otherwise} \end{cases}$$

Learning Laws: L1-L2 and L2-L1



The ART1 network has two separate learning laws: one for the L1-L2 connections (instars) and one for the L2-L1 connections (outstars).

Both sets of connections are updated at the same time - when the input and the expectation have an adequate match.

The process of matching, and subsequent adaptation is referred to as resonance.

Subset/Superset Dilemma

Suppose that $\mathbf{W}^{1:2} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ so the prototypes are $_1\mathbf{w}^{1:2} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ $_2\mathbf{w}^{1:2} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

We say that $_1\mathbf{w}^{1:2}$ is a subset of $_2\mathbf{w}^{1:2}$, because $_2\mathbf{w}^{1:2}$ has a 1 wherever $_1\mathbf{w}^{1:2}$ has a 1.

If the output of layer 1 is $\mathbf{a}^1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ then the input to Layer 2 will be

$$\mathbf{W}^{1:2}\mathbf{a}^1 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

Both prototype vectors have the same inner product with $\mathbf{a}^1$, even though the first prototype is identical to $\mathbf{a}^1$ and the second prototype is not. This is called the *Subset/Superset* dilemma.

Subset/Superset Solution

Normalize the prototype patterns.

$$\mathbf{W}^{1:2} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

$$\mathbf{W}^{1:2}\mathbf{a}^{1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{2}{3} \end{bmatrix}$$

Now we have the desired result; the first prototype has the largest inner product with the input.

L1-L2 Learning Law

Instar Learning with Competition

$$\frac{d[{}_i\mathbf{w}^{1:2}(t)]}{dt} = a_i^2(t)[\{{}^+\mathbf{b} - {}_i\mathbf{w}^{1:2}(t)\}\zeta[{}^+\mathbf{W}]\mathbf{a}^1(t) - \{{}_i\mathbf{w}^{1:2}(t) + {}^-\mathbf{b}\}[{}^-\mathbf{W}]\mathbf{a}^1(t)] ,$$

where

$${}^+\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \quad {}^-\mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad {}^+\mathbf{W} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \quad {}^-\mathbf{W} = \begin{bmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 1 \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \cdots & 0 \end{bmatrix}$$

Upper Limit Bias — Lower Limit Bias — On-Center Connections — Off-Surround Connections

When neuron i of Layer 2 is active, ${}_i\mathbf{w}^{1:2}$ is moved in the direction of $\mathbf{a}^1$. The elements of ${}_i\mathbf{w}^{1:2}$ compete, and therefore ${}_i\mathbf{w}^{1:2}$ is normalized.

Fast Learning

$$\frac{dw^{1:2}_{i,j}(t)}{dt} = a^2_i(t)\left[(1 - w^{1:2}_{i,j}(t))\zeta a^1_j(t) - w^{1:2}_{i,j}(t)\sum_{k \neq j} a^1_k(t)\right]$$

For *fast learning* we assume that the outputs of Layer 1 and Layer 2 remain constant until the weights reach steady state.

Assume that $a^2_i(t) = 1$, and solve for the steady state weight:

$$0 = \left[(1 - w^{1:2}_{i,j})\zeta a^1_j - w^{1:2}_{i,j}\sum_{k \neq j} a^1_k\right]$$

Case I: $a^1_j = 1$

$$\left.0 = (1 - w^{1:2}_{i,j})\zeta - w^{1:2}_{i,j}(\|\mathbf{a}^1\|^2 - 1) = -(\zeta + \|\mathbf{a}^1\|^2 - 1)w^{1:2}_{i,j} + \zeta \right\} \quad w^{1:2}_{i,j} = \frac{\zeta}{\zeta + \|\mathbf{a}^1\|^2 - 1}$$

Case II: $a^1_j = 0$

$$\left.0 = -w^{1:2}_{i,j}\|\mathbf{a}^1\|^2 \right\} \quad w^{1:2}_{i,j} = 0$$

Summary

$${}_i\mathbf{w}^{1:2} = \frac{\zeta \mathbf{a}^1}{\zeta + \|\mathbf{a}^1\|^2 - 1}$$

19

Learning Law: L2-L1

Outstar

$$\frac{d[\mathbf{w}_j^{2:1}(t)]}{dt} = a_j^2(t)[-\mathbf{w}_j^{2:1}(t) + \mathbf{a}^1(t)]$$

Fast Learning

Assume that $a^2_j(t) = 1$, and solve for the steady state weight:

$$\mathbf{0} = -\mathbf{w}_j^{2:1} + \mathbf{a}^1 \quad \text{or} \quad \mathbf{w}_j^{2:1} = \mathbf{a}^1$$

Column j of $\mathbf{W}^{2:1}$ converges to the output of Layer 1, which is a combination of the input pattern and the previous prototype pattern. The prototype pattern is modified to incorporate the current input pattern.

ART1 Algorithm Summary

0) All elements of the initial $\mathbf{W}^{2:1}$ matrix are set to 1. All elements of the initial $\mathbf{W}^{1:2}$ matrix are set to $\zeta/(\zeta+S^1-1)$.
1) Input pattern is presented. Since Layer 2 is not active,

$$\mathbf{a}^1 = \mathbf{p}$$

2) The input to Layer 2 is computed, and the neuron with the largest input is activated.

$$a_i^2 = \begin{cases} 1, & \text{if}(({}_i\mathbf{w}^{1:2})^T \mathbf{a}^1 = max[({}_k\mathbf{w}^{1:2})^T \mathbf{a}^1]) \\ 0, & \text{otherwise} \end{cases}$$

In case of a tie, the neuron with the smallest index is the winner.

3) The L2-L1 expectation is computed.

$$\mathbf{W}^{2:1}\mathbf{a}^2 = \mathbf{w}_j^{2:1}$$

Summary Continued

4) Layer 1 output is adjusted to include the L2-L1 expectation.

$$\mathbf{a}^1 = \mathbf{p} \cap \mathbf{w}_j^{2:1}$$

5) The orienting subsystem determines match between the expectation and the input pattern.

$$a^0 = \begin{cases} 1, & \text{if}\left[\left\|\mathbf{a}^1\right\|^2 / \left\|\mathbf{p}\right\|^2 < \rho\right] \\ 0, & \text{otherwise} \end{cases}$$

6) If $a^0 = 1$, then set $a^2_j = 0$, inhibit it until resonance, and return to Step 1. If $a^0 = 0$, then continue with Step 7.
7) Resonance has occured. Update row j of $\mathbf{W}^{1:2}$.

$${}_j\mathbf{w}^{1:2} = \frac{\zeta \mathbf{a}^1}{\zeta + \left\|\mathbf{a}^1\right\|^2 - 1}$$

8) Update column j of $\mathbf{W}^{2:1}$.

$$\mathbf{w}_j^{2:1} = \mathbf{a}^1$$

9) Remove input, restore inhibited neurons, and return to Step 1.