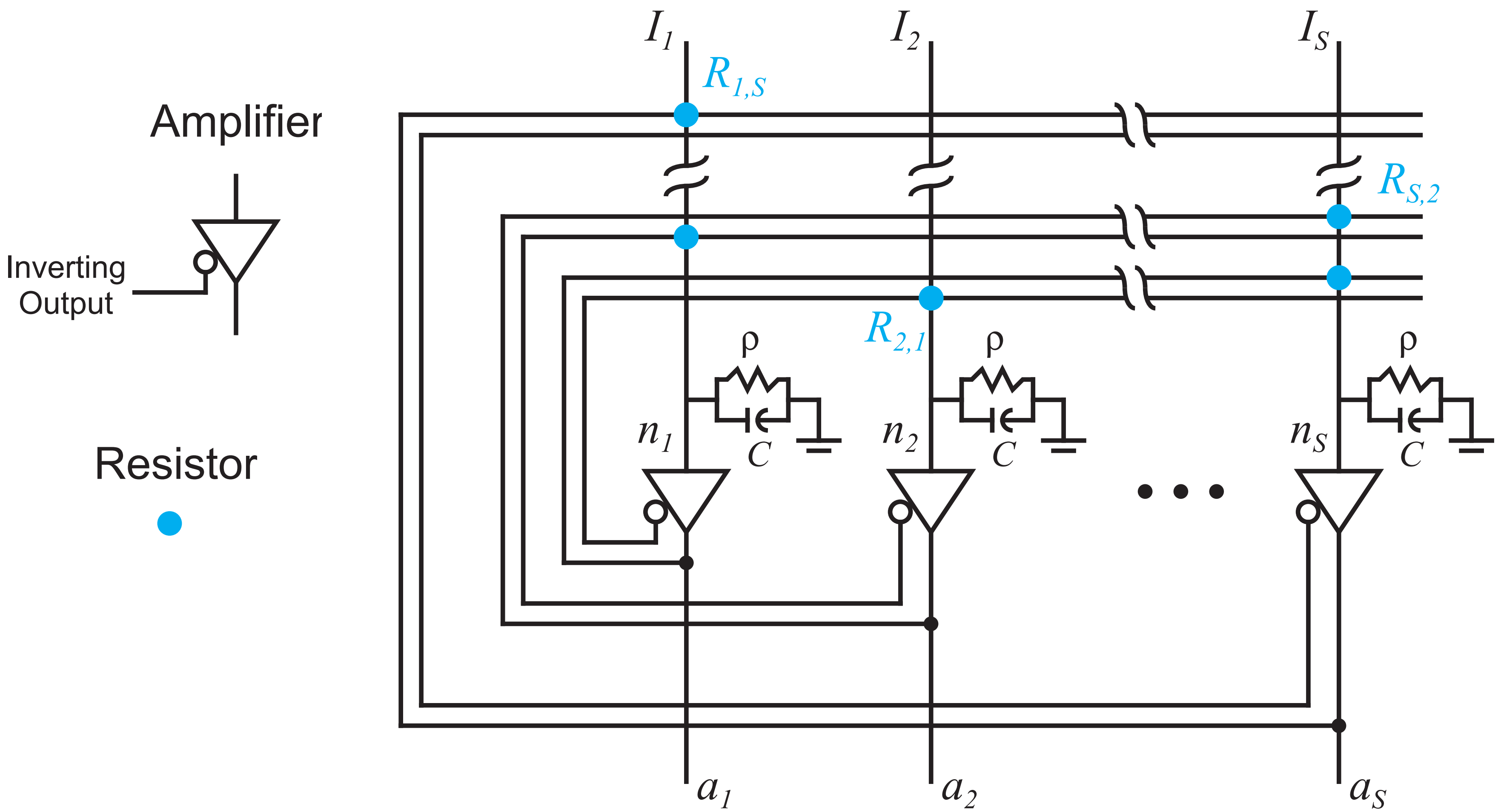
Hopfield Network

Hopfield Model



Equations of Operation

$$C\frac{dn_i(t)}{dt} = \sum_{j=1}^{S} T_{i,j}a_j(t) - \frac{n_i(t)}{R_i} + I_i$$

n_i - input voltage to the *i*th amplifier
a_i - output voltage of the *i*th amplifier
C - amplifier input capacitance
I_i - fixed input current to the *i*th amplifier

$$|T_{i,j}| = \frac{1}{R_{i,j}} \qquad \frac{1}{R_i} = \frac{1}{\rho} + \sum_{j=1}^{S}\frac{1}{R_{i,j}} \qquad n_i = f^{-1}(a_i) \qquad a_i = f(n_i)$$

Network Format

$$R_i C \frac{dn_i(t)}{dt} = \sum_{j=1}^{S} R_i T_{i,j} a_j(t) - n_i(t) + R_i I_i$$

Define:

$$\varepsilon = R_i C \qquad w_{i,j} = R_i T_{i,j} \qquad b_i = R_i I_i$$

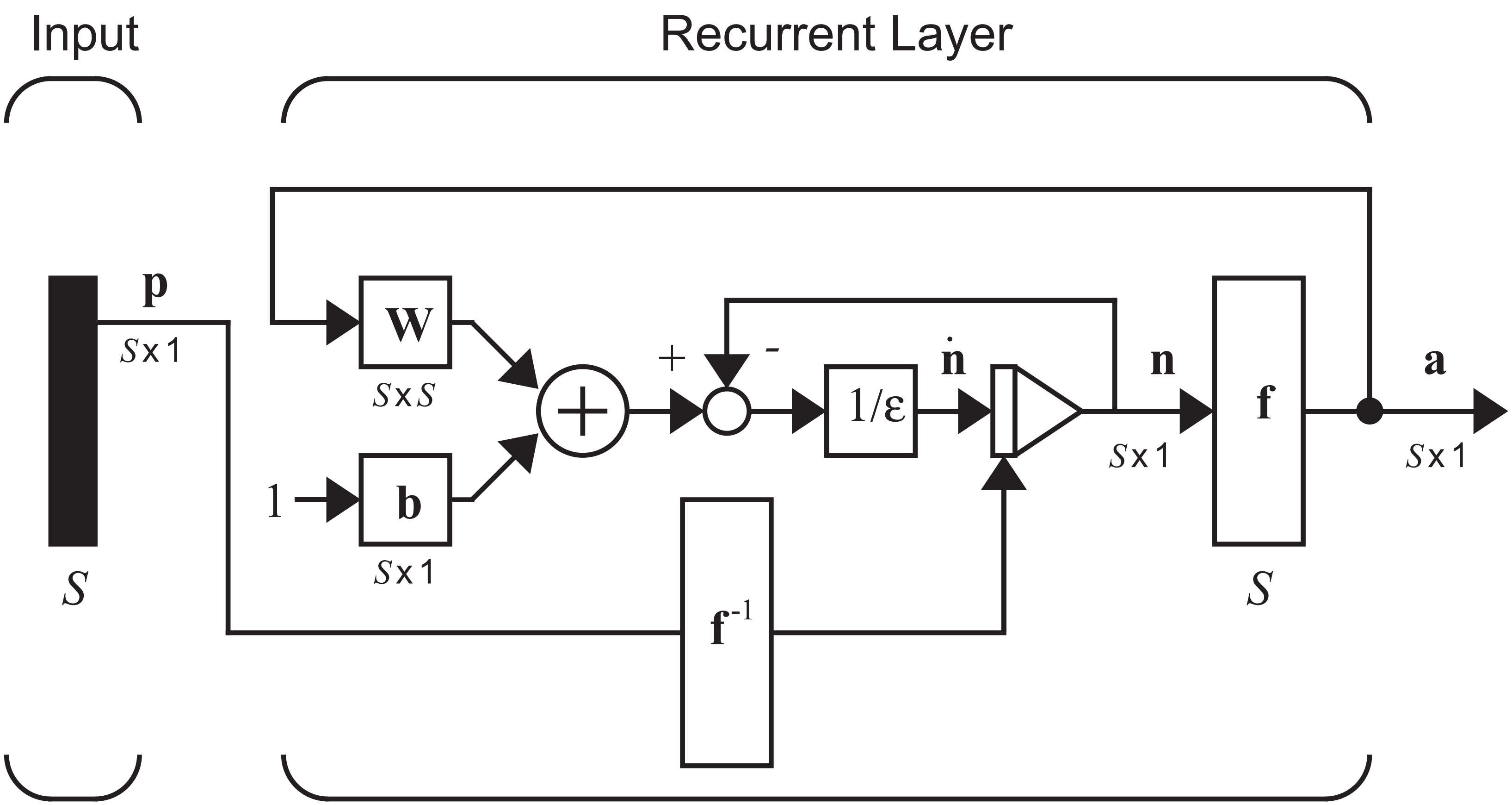
$$\varepsilon \frac{dn_i(t)}{dt} = -n_i(t) + \sum_{j=1}^{S} w_{i,j} a_j(t) + b_i$$

Vector Form:

$$\varepsilon \frac{d\mathbf{n}(t)}{dt} = -\mathbf{n}(t) + \mathbf{W}\mathbf{a}(t) + \mathbf{b}$$

$$\mathbf{a}(t) = \mathbf{f}(\mathbf{n}(t))$$

Hopfield Network



$$\mathbf{n}(0) = \mathbf{f}^{-1}(\mathbf{p}), \quad (\mathbf{a}(0) = \mathbf{p}) \qquad \varepsilon\, d\mathbf{n}/dt = -\mathbf{n} + \mathbf{W}\,\mathbf{f}(\mathbf{n}) + \mathbf{b}$$

Lyapunov Function

$$V(\mathbf{a}) = -\frac{1}{2}\mathbf{a}^T\mathbf{W}\mathbf{a} + \sum_{i=1}^{S}\left\{\int_0^{a_i} f^{-1}(u)\,du\right\} - \mathbf{b}^T\mathbf{a}$$

Individual Derivatives

First Term:

$$\frac{d}{dt}\left\{-\frac{1}{2}\mathbf{a}^T\mathbf{W}\mathbf{a}\right\} = -\frac{1}{2}\nabla[\mathbf{a}^T\mathbf{W}\mathbf{a}]^T\frac{d\mathbf{a}}{dt} = -[\mathbf{W}\mathbf{a}]^T\frac{d\mathbf{a}}{dt} = -\mathbf{a}^T\mathbf{W}\frac{d\mathbf{a}}{dt}$$

Second Term:

$$\frac{d}{dt}\left\{\int_0^{a_i} f^{-1}(u)\,du\right\} = \frac{d}{da_i}\left\{\int_0^{a_i} f^{-1}(u)\,du\right\}\frac{da_i}{dt} = f^{-1}(a_i)\frac{da_i}{dt} = n_i\frac{da_i}{dt}$$

$$\frac{d}{dt}\left[\sum_{i=1}^{S}\left\{\int_0^{a_i} f^{-1}(u)\,du\right\}\right] = \mathbf{n}^T\frac{d\mathbf{a}}{dt}$$

Third Term:

$$\frac{d}{dt}\{-\mathbf{b}^T\mathbf{a}\} = -\nabla[\mathbf{b}^T\mathbf{a}]^T\frac{d\mathbf{a}}{dt} = -\mathbf{b}^T\frac{d\mathbf{a}}{dt}$$

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Complete Lyapunov Derivative

$$\frac{d}{dt}V(\mathbf{a}) = -\mathbf{a}^T\mathbf{W}\frac{d\mathbf{a}}{dt} + \mathbf{n}^T\frac{d\mathbf{a}}{dt} - \mathbf{b}^T\frac{d\mathbf{a}}{dt} = [-\mathbf{a}^T\mathbf{W} + \mathbf{n}^T - \mathbf{b}^T]\frac{d\mathbf{a}}{dt}$$

From the system equations we know:

$$[-\mathbf{a}^T\mathbf{W} + \mathbf{n}^T - \mathbf{b}^T] = -\varepsilon\left[\frac{d\mathbf{n}(t)}{dt}\right]^T$$

So the derivative can be written:

$$\frac{d}{dt}V(\mathbf{a}) = -\varepsilon\left[\frac{d\mathbf{n}(t)}{dt}\right]^T\frac{d\mathbf{a}}{dt} = -\varepsilon\sum_{i=1}^{S}\left(\frac{dn_i}{dt}\right)\left(\frac{da_i}{dt}\right) = -\varepsilon\sum_{i=1}^{S}\left(\frac{dn_i}{dt}\right)\left(\frac{da_i}{dt}\right)$$

$$= -\varepsilon\sum_{i=1}^{S}\left(\frac{d}{da_i}[f^{-1}(a_i)]\right)\left(\frac{da_i}{dt}\right)^2$$

If $\frac{d}{da_i}[f^{-1}(a_i)] > 0$ then $\frac{d}{dt}V(\mathbf{a}) \leq 0$

Invariant Sets

$$Z = \{\mathbf{a}: dV(\mathbf{a})/dt = 0,\ \mathbf{a} \text{ in the closure of } G\}$$

$$\frac{d}{dt}V(\mathbf{a}) = -\varepsilon \sum_{i=1}^{S} \left(\frac{d}{da_i}[f^{-1}(a_i)]\right)\left(\frac{da_i}{dt}\right)^2$$

This will be zero only if the neuron outputs are not changing:

$$\frac{d\mathbf{a}}{dt} = \mathbf{0}$$

Therefore, the system energy is not changing only at the equilibrium points of the circuit. Thus, all points in Z are potential attractors:

$$L = Z$$

Example

$$a = f(n) = \frac{2}{\pi}\tan^{-1}\left(\frac{\gamma\pi n}{2}\right) \qquad n = \frac{2}{\gamma\pi}\tan\left(\frac{\pi}{2}a\right)$$

$$\left.\begin{array}{l} R_{1,2} = R_{2,1} = 1 \\ \\ T_{1,2} = T_{2,1} = 1 \end{array}\right\} \quad \mathbf{W} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\varepsilon = R_i C = 1$$

$$\gamma = 1.4$$

$$\left. I_1 = I_2 = 0 \right\} \quad \mathbf{b} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

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Example Lyapunov Function

$$V(\mathbf{a}) = -\frac{1}{2}\mathbf{a}^T\mathbf{W}\mathbf{a} + \sum_{i=1}^{S}\left\{\int_0^{a_i} f^{-1}(u)\,du\right\} - \mathbf{b}^T\mathbf{a}$$

$$-\frac{1}{2}\mathbf{a}^T\mathbf{W}\mathbf{a} = -\frac{1}{2}\begin{bmatrix} a_1 & a_2 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = -a_1 a_2$$

$$\int_0^{a_i} f^{-1}(u)\,du = \frac{2}{\gamma\pi}\int_0^{a_i} \tan\left(\frac{\pi}{2}u\right)du = \frac{2}{\gamma\pi}\left[-\log\left[\cos\left(\frac{\pi}{2}u\right)\right]\frac{2}{\pi}\right]_0^{a_i} = -\frac{4}{\gamma\pi^2}\log\left[\cos\left(\frac{\pi}{2}a_i\right)\right]$$

$$V(\mathbf{a}) = -a_1 a_2 - \frac{4}{1.4\pi^2}\left[\log\left\{\cos\left(\frac{\pi}{2}a_1\right)\right\} + \log\left\{\cos\left(\frac{\pi}{2}a_2\right)\right\}\right]$$

Example Network Equations

$$\frac{d\mathbf{n}}{dt} = -\mathbf{n} + \mathbf{W}\mathbf{f}(\mathbf{n}) = -\mathbf{n} + \mathbf{W}\mathbf{a}$$

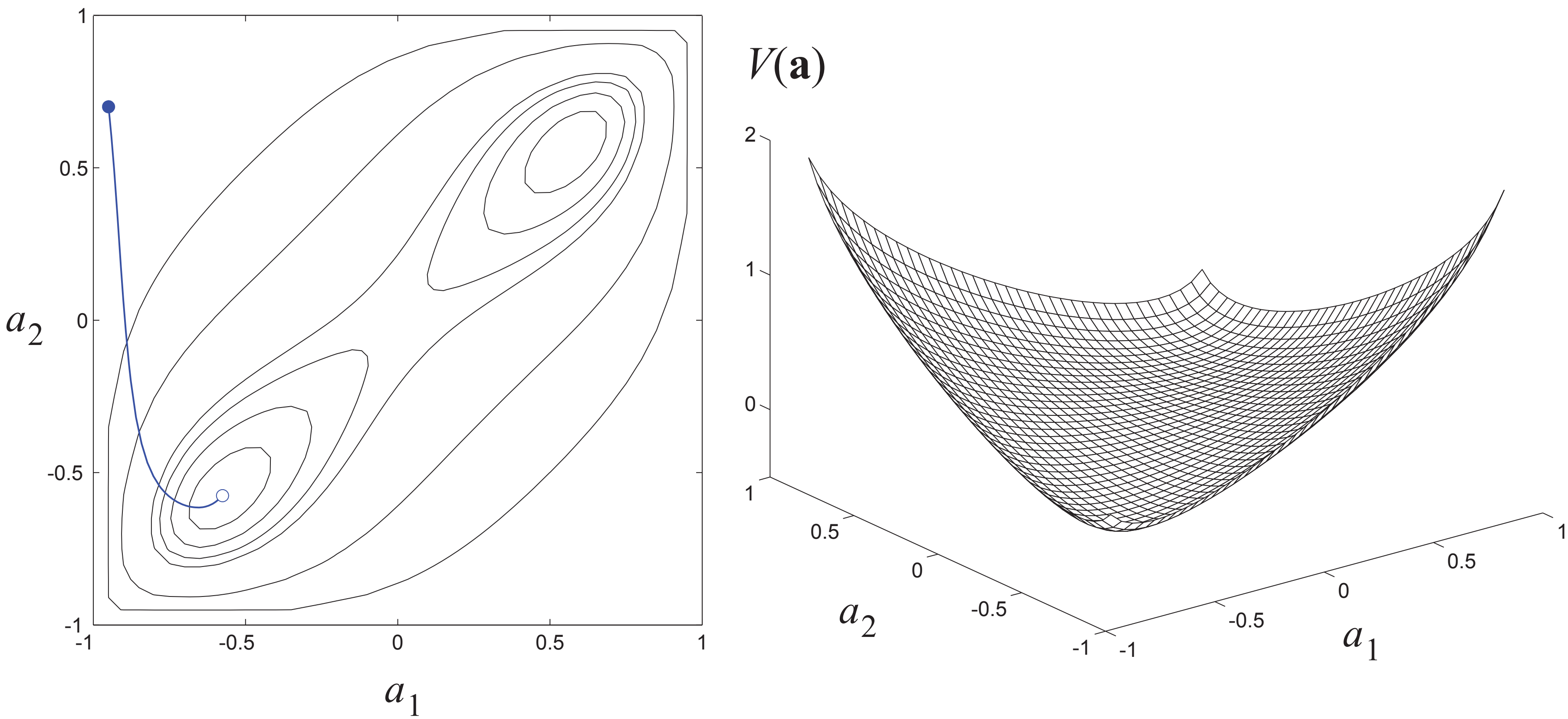
$$dn_1/dt = a_2 - n_1$$

$$dn_2/dt = a_1 - n_2$$

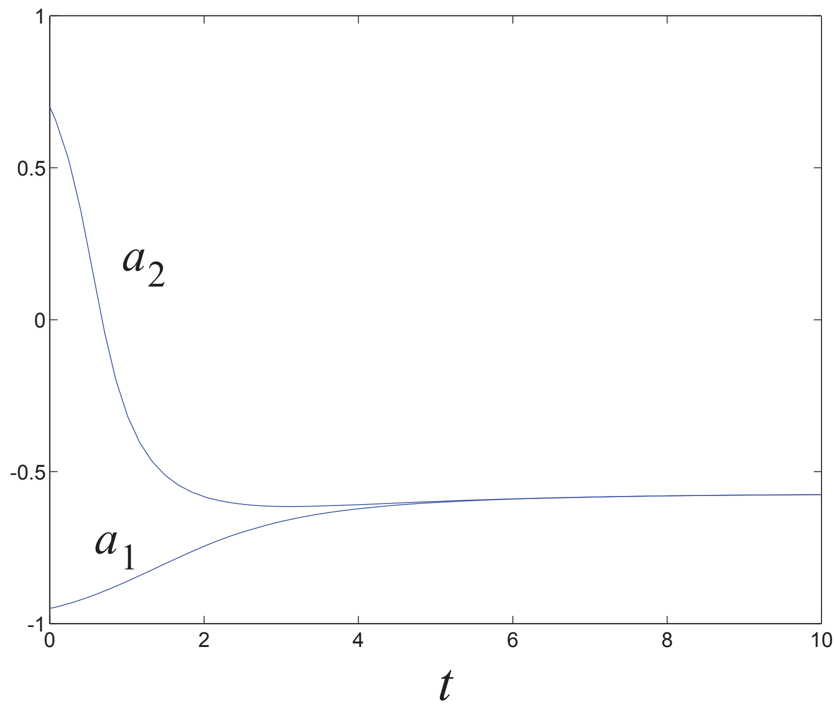
$$a_1 = \frac{2}{\pi}\tan^{-1}\left(\frac{1.4\pi}{2}n_1\right)$$

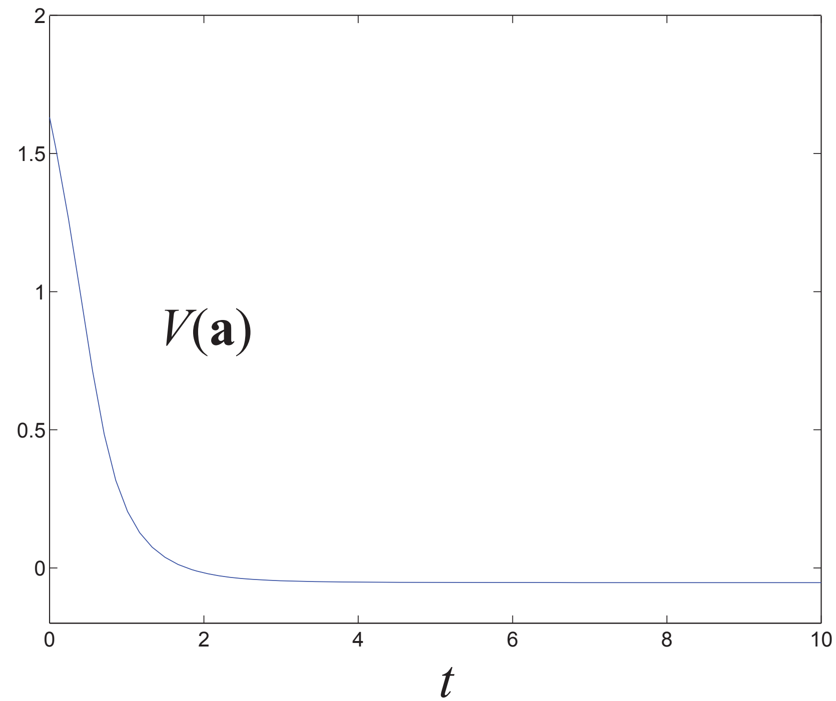
$$a_2 = \frac{2}{\pi}\tan^{-1}\left(\frac{1.4\pi}{2}n_2\right)$$

21 Lyapunov Function and Trajectory

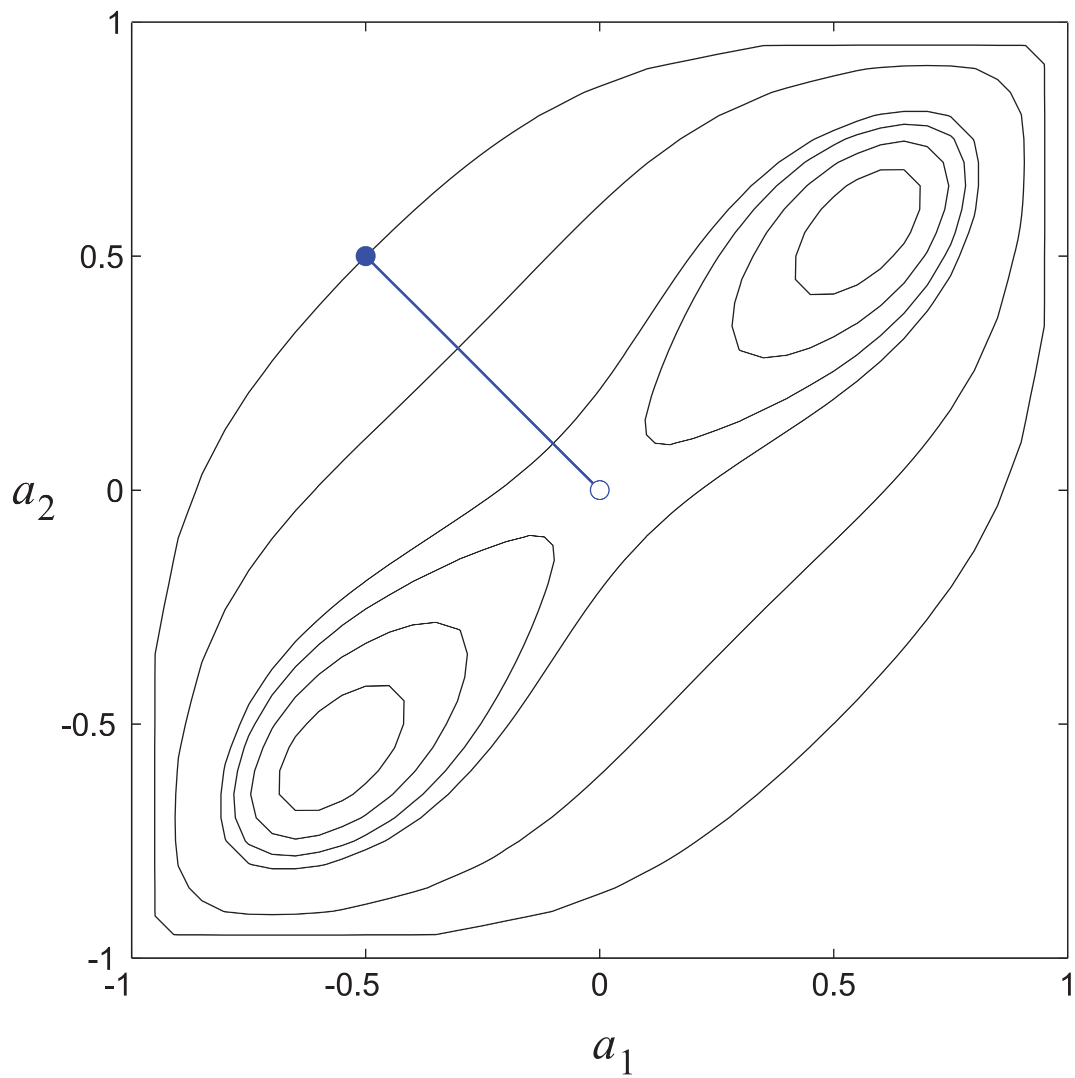


Time Response





Convergence to a Saddle Point



Hopfield Attractors

The potential attractors of the Hopfield network satisfy:

$$\frac{d\mathbf{a}}{dt} = \mathbf{0}$$

How are these points related to the minima of $V(\mathbf{a})$? The minima must satisfy:

$$\nabla V = \left[\frac{\partial V}{\partial a_1}\ \frac{\partial V}{\partial a_2}\ \cdots\ \frac{\partial V}{\partial a_S}\right]^T = \mathbf{0}$$

Where the Lyapunov function is given by:

$$V(\mathbf{a}) = -\frac{1}{2}\mathbf{a}^T\mathbf{W}\mathbf{a} + \sum_{i=1}^{S}\left\{\int_0^{a_i} f^{-1}(u)du\right\} - \mathbf{b}^T\mathbf{a}$$

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Hopfield Attractors

Using previous results, we can show that:

$$\nabla V(\mathbf{a}) = [-\mathbf{W}\mathbf{a} + \mathbf{n} - \mathbf{b}] = -\varepsilon\left[\frac{d\mathbf{n}(t)}{dt}\right]$$

The *i*th element of the gradient is therefore:

$$\frac{\partial}{\partial a_i} V(\mathbf{a}) = -\varepsilon \frac{dn_i}{dt} = -\varepsilon \frac{d}{dt}([f^{-1}(a_i)]) = -\varepsilon \frac{d}{d\,a_i}[f^{-1}(a_i)]\frac{da_i}{d\,t}$$

Since the transfer function and its inverse are monotonic increasing:

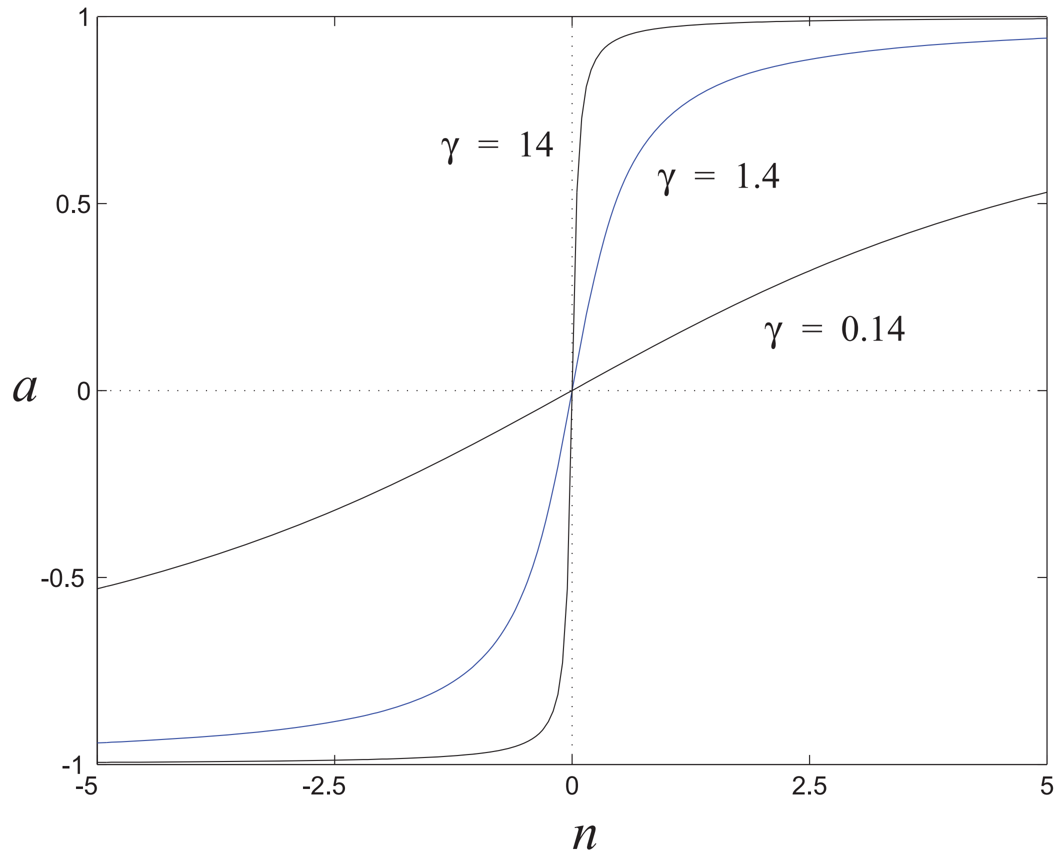
$$\frac{d}{da_i}[f^{-1}(a_i)] > 0$$

All points for which $\frac{d\mathbf{a}(t)}{dt} = \mathbf{0}$ will also satisfy $\nabla V(\mathbf{a}) = \mathbf{0}$

Therefore all attractors will be stationary points of $V(\mathbf{a})$.

Effect of Gain

$$a = f(n) = \frac{2}{\pi}\tan^{-1}\left(\frac{\gamma\pi n}{2}\right)$$



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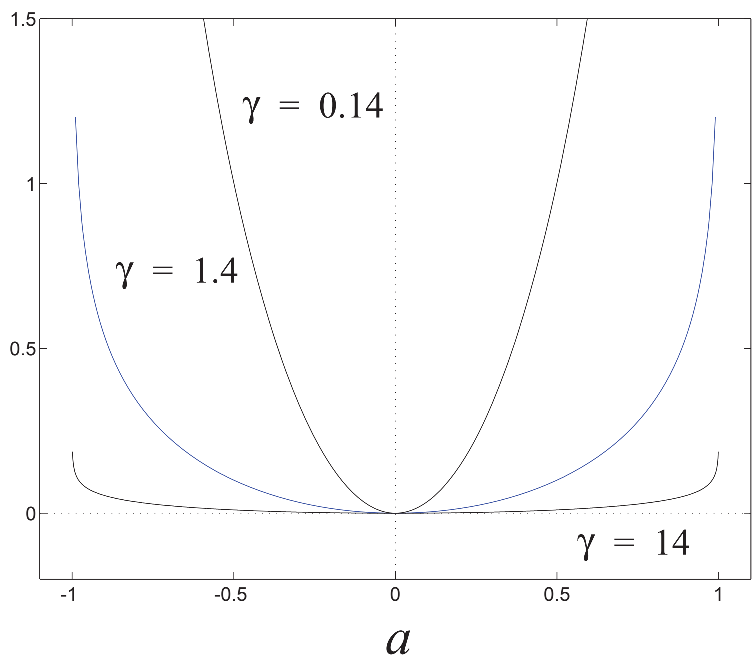
Lyapunov Function

$$V(\mathbf{a}) = -\frac{1}{2}\mathbf{a}^T\mathbf{W}\mathbf{a} + \sum_{i=1}^{S}\left\{\int_0^{a_i} f^{-1}(u)\,du\right\} - \mathbf{b}^T\mathbf{a}$$

$$f^{-1}(u) = \frac{2}{\gamma\pi}\tan\left(\frac{\pi u}{2}\right)$$

$$\int_0^{a_i} f^{-1}(u)\,du = \frac{2}{\gamma\pi}\left[\frac{2}{\pi}\log\left(\cos\left(\frac{\pi a_i}{2}\right)\right)\right] = -\frac{4}{\gamma\pi^2}\log\left[\cos\left(\frac{\pi a_i}{2}\right)\right]$$

$$-\frac{4}{\gamma\pi^2}\log\left[\cos\left(\frac{\pi a}{2}\right)\right]$$



21 High Gain Lyapunov Function

As $\gamma \rightarrow \infty$ the Lyapunov function reduces to:

$$V(\mathbf{a}) = -\frac{1}{2}\mathbf{a}^T\mathbf{W}\mathbf{a} - \mathbf{b}^T\mathbf{a}$$

The high gain Lyapunov function is quadratic:

$$V(\mathbf{a}) = -\frac{1}{2}\mathbf{a}^T\mathbf{W}\mathbf{a} - \mathbf{b}^T\mathbf{a} = \frac{1}{2}\mathbf{a}^T\mathbf{A}\mathbf{a} + \mathbf{d}^T\mathbf{a} + c$$

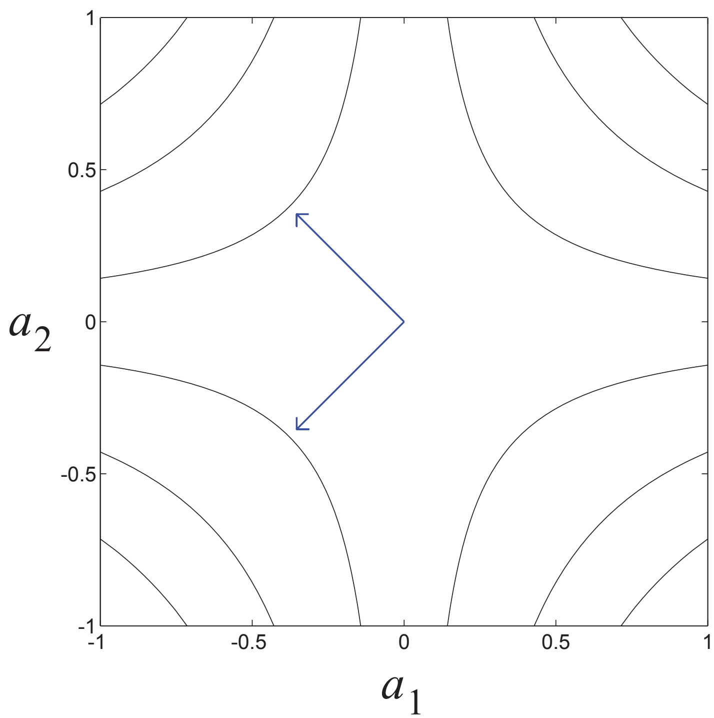
where

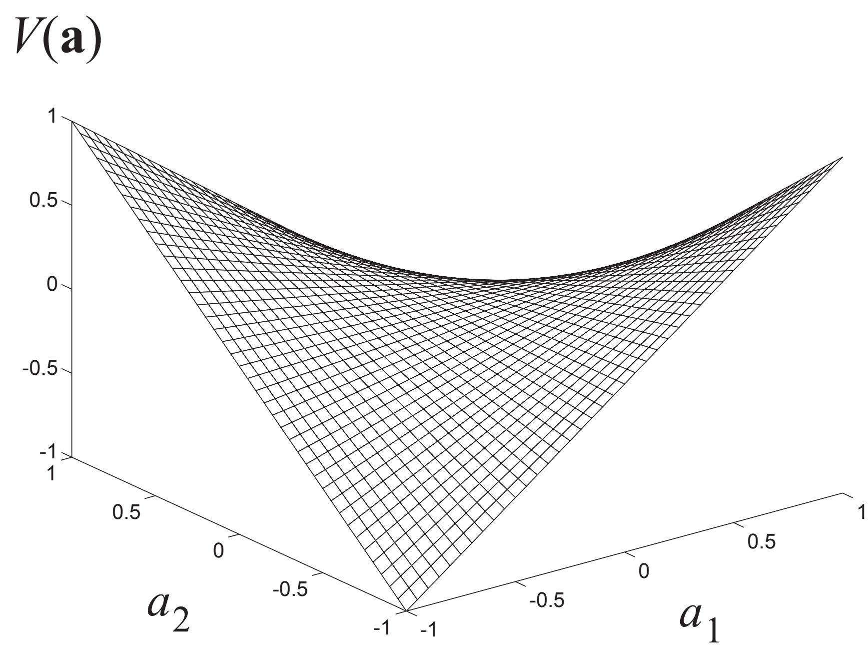
$$\nabla^2 V(\mathbf{a}) = \mathbf{A} = -\mathbf{W} \qquad \mathbf{d} = -\mathbf{b} \qquad c = 0$$

Example

$$\nabla^2 V(\mathbf{a}) = -\mathbf{W} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \qquad \left|\nabla^2 V(\mathbf{a}) - \lambda \mathbf{I}\right| = \begin{vmatrix} -\lambda & -1 \\ -1 & -\lambda \end{vmatrix} = \lambda^2 - 1 = (\lambda + 1)(\lambda - 1)$$

$$\lambda_1 = -1 \qquad \mathbf{z}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad\qquad \lambda_2 = 1 \qquad \mathbf{z}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$





Hopfield Design

The Hopfield network will minimize the following Lyapunov function:

$$V(\mathbf{a}) = -\frac{1}{2}\mathbf{a}^T\mathbf{W}\mathbf{a} - \mathbf{b}^T\mathbf{a}$$

Choose the weight matrix **W** and the bias vector **b** so that V takes on the form of a function you want to minimize.

21 Content-Addressable Memory

Content-Addressable Memory - retrieves stored memories on the basis of part of the contents.

Prototype Patterns:

$$\{\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_Q\} \qquad \text{(bipolar vectors)}$$

Proposed Performance Index:

$$J(\mathbf{a}) = -\frac{1}{2}\sum_{q=1}^{Q}\left([\mathbf{p}_q]^T\mathbf{a}\right)^2$$

For orthogonal prototypes, if we evaluate the performance index at a prototype:

$$J(\mathbf{p}_j) = -\frac{1}{2}\sum_{q=1}^{Q}\left([\mathbf{p}_q]^T\mathbf{p}_j\right)^2 = -\frac{1}{2}\left([\mathbf{p}_j]^T\mathbf{p}_j\right)^2 = -\frac{S}{2}$$

$J(\mathbf{a})$ will be largest when **a** is not close to any prototype pattern, and smallest when **a** is equal to a prototype pattern.

Hebb Rule

If we use the supervised Hebb rule to compute the weight matrix:

$$\mathbf{W} = \sum_{q=1}^{Q} \mathbf{p}_q (\mathbf{p}_q)^T \qquad \mathbf{b} = \mathbf{0}$$

the Lyapunov function will be:

$$V(\mathbf{a}) = -\frac{1}{2}\mathbf{a}^T\mathbf{W}\mathbf{a} = -\frac{1}{2}\mathbf{a}^T\left[\sum_{q=1}^{Q} \mathbf{p}_q (\mathbf{p}_q)^T\right]\mathbf{a} = -\frac{1}{2}\sum_{q=1}^{Q} \mathbf{a}^T\mathbf{p}_q (\mathbf{p}_q)^T\mathbf{a}$$

This can be rewritten:

$$V(\mathbf{a}) = -\frac{1}{2}\sum_{q=1}^{Q} [(\mathbf{p}_q)^T\mathbf{a}]^2 = J(\mathbf{a})$$

Therefore the Lyapunov function is equal to our performance index for the content addressable memory.

Hebb Rule Analysis

$$\mathbf{W} = \sum_{q=1}^{Q} \mathbf{p}_q (\mathbf{p}_q)^T$$

If we apply prototype $\mathbf{p}_j$ to the network:

$$\mathbf{W}\mathbf{p}_j = \sum_{q=1}^{Q} \mathbf{p}_q (\mathbf{p}_q)^T \mathbf{p}_j = \mathbf{p}_j (\mathbf{p}_j)^T \mathbf{p}_j = S\mathbf{p}_j$$

Therefore each prototype is an eigenvector, and they have a common eigenvalue of S. The eigenspace for the eigenvalue $\lambda = S$ is therefore:

$$X = \mathrm{span}\{\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_Q\}$$

An S-dimensional space of all vectors which can be written as linear combinations of the prototype vectors.

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Weight Matrix Eigenspace

The entire input space can be divided into two disjoint sets:

$$R^S = X \cup X^{\perp}$$

where $X^{\perp}$ is the orthogonal complement of X. For vectors **a** in the orthogonal complement we have:

$$(\mathbf{p}_q)^T \mathbf{a} = 0, \quad q = 1, 2, \ldots, Q$$

Therefore,

$$\mathbf{W}\mathbf{a} = \sum_{q=1}^{Q} \mathbf{p}_q (\mathbf{p}_q)^T \mathbf{a} = \sum_{q=1}^{Q} (\mathbf{p}_q \times 0) = \mathbf{0} = 0 \times \mathbf{a}$$

The eigenvalues of **W** are S and 0, with corresponding eigenspaces of X and $X^{\perp}$. For the Hessian matrix

$$\nabla^2 V = -\mathbf{W}$$

the eigenvalues are $-S$ and 0, with the same eigenspaces.

Lyapunov Surface

The high-gain Lyapunov function is a quadratic function. Therefore, the eigenvalues of the Hessian matrix determine its shape. Because the first eigenvalue is negative, V will have negative curvature in X. Because the second eigenvalue is zero, V will have zero curvature in $X^{\perp}$.

Because V has negative curvature in X, the trajectories of the Hopfield network will tend to fall into the corners of the hypercube $\{\mathbf{a}: -1 < a_i < 1\}$ that are contained in X.

Example

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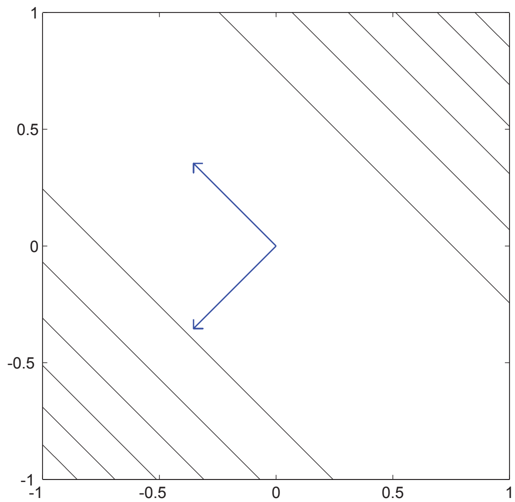
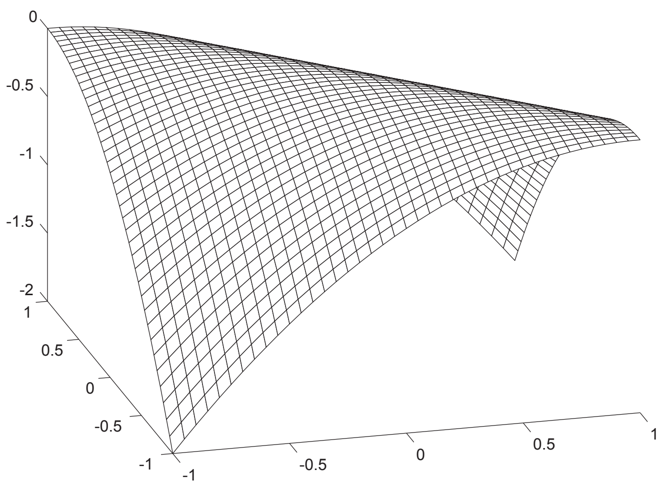
$$\mathbf{p}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \mathbf{W} = \mathbf{p}_1(\mathbf{p}_1)^T = \begin{bmatrix} 1 \\ 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \qquad V(\mathbf{a}) = -\frac{1}{2}\mathbf{a}^T\mathbf{W}\mathbf{a} = -\frac{1}{2}\mathbf{a}^T\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\mathbf{a}$$



$$\nabla^2 V(\mathbf{a}) = -\mathbf{W} = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}$$

$$\lambda_1 = -S = -2 \qquad \mathbf{z}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$X = \{\mathbf{a} : a_1 = a_2\}$$



$$\lambda_2 = 0 \qquad \mathbf{z}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$X^{\perp} = \{\mathbf{a} : a_1 = -a_2\}$$

21 Zero Diagonal Elements

We can zero the diagonal elements of the weight matrix:

$$\mathbf{W}' = \mathbf{W} - Q\mathbf{I}$$

The prototypes remain eigenvectors of this new matrix, but the corresponding eigenvalue is now (S-Q):

$$\mathbf{W}'\mathbf{p}_q = [\mathbf{W} - Q\mathbf{I}]\mathbf{p}_q = S\mathbf{p}_q - Q\mathbf{p}_q = (S - Q)\mathbf{p}_q$$

The elements of $X^{\perp}$ also remain eigenvectors of this new matrix, with a corresponding eigenvalue of (-Q):

$$\mathbf{W}'\mathbf{a} = [\mathbf{W} - Q\mathbf{I}]\mathbf{a} = \mathbf{0} - Q\mathbf{a} = -Q\mathbf{a}$$

The Lyapunov surface will have negative curvature in X and positive curvature in $X^{\perp}$, in contrast with the original Lyapunov function, which had negative curvature in X and zero curvature in $X^{\perp}$.

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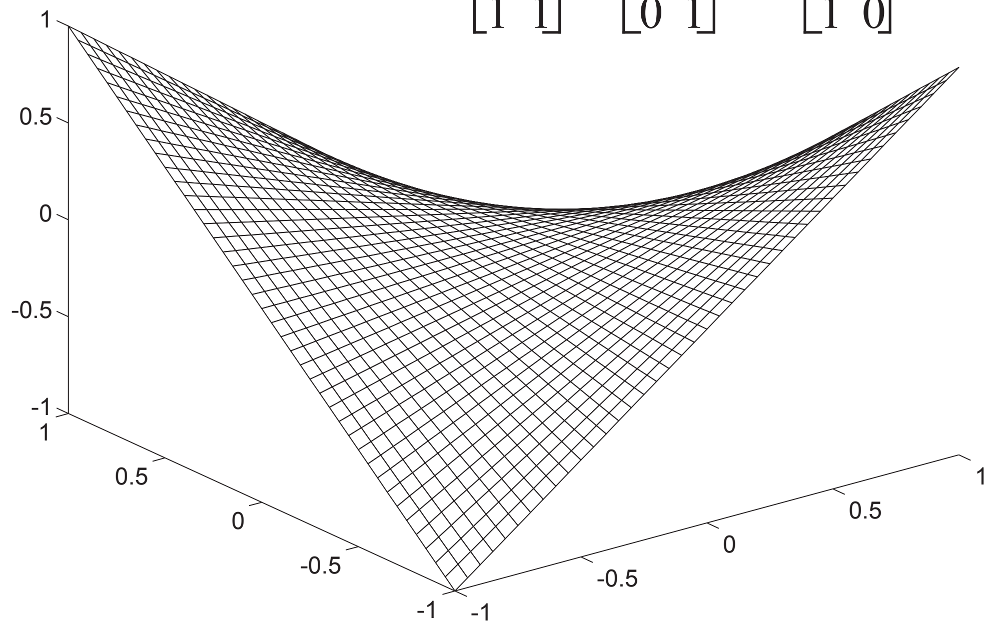
Example

$$\mathbf{W}' = \mathbf{W} - Q\mathbf{I} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$



If the initial condition falls exactly on the line a_1=-a_2, and the weight matrix **W** is used, then the network output will remain constant. If the initial condition falls exactly on the line a_1=-a_2, and the weight matrix **W**' is used, then the network output will converge to the saddle point at the origin.