CSE 3318 Notes 3: Summations

(Last updated 1/5/24 10:26 AM)

CLRS, appendix A

3.A. Geometric Series (review)

$$\sum_{k=0}^{t} x^k = \frac{x^{t+1}-1}{x-1} \qquad \text{when } x \neq 1 \qquad \text{[Not hard to verify by math induction]}$$

$$\sum_{k=0}^{t} x^k \leq \sum_{k=0}^{\infty} x^k = \lim_{k\to\infty} \frac{x^k-1}{x-1} = \frac{1}{1-x} \qquad \text{when } 0 < x < 1$$

3.B. Harmonic Series

$$\ln n \leq H_n = \sum_{k=1}^{n} \frac{1}{k} \leq \ln n + .577\ldots \leq \ln n + 1$$

3.C. Approximation by Integrals (p. 1150-1151)

For a monotonically increasing $(x \leq y \Rightarrow f(x) \leq f(y))$ function:

$$\int_{m-1}^{n} f(x)dx \leq \sum_{k=m}^{n} f(k) \leq \int_{m}^{n+1} f(x)dx$$

Since:

$$\int_{k-1}^{k} f(x)dx \leq f(k) \leq \int_{k}^{k+1} f(x)dx$$

in this situation.

3.D. Bounding Summations Using Math Induction and Inequalities

[Techniques are especially important for recurrences in Notes 04]

Show $\sum_{i=1}^{n} i^2 = \Theta\left(n^3\right)$ [Trivial to show using integration or $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.]

a. Show O(n^3)

(i) $\sum_{i=1}^{n=1} i^2 = 1 \le cn^3$ using any constant $c \ge 1$

(ii) Suppose this holds for n:

$$\sum_{i=1}^{n} i^2 \le cn^3$$

Now go on to $n + 1$ and show that the bound *still holds*

$$\begin{aligned}\sum_{i=1}^{n+1} i^2 &= \sum_{i=1}^{n} i^2 + (n+1)^2 \\ &= \sum_{i=1}^{n} i^2 + n^2 + 2n + 1 \\ &\le cn^3 + n^2 + 2n + 1 \\ &= ??? \\ &\le c(n+1)^3\end{aligned}$$

The bridging step (???) separates the bounding term ($c(n+1)^3$) from everything else (x):

$$c(n+1)^3 + x = cn^3 + n^2 + 2n + 1$$

$$x = cn^3 + n^2 + 2n + 1 - cn^3 - 3cn^2 - 3cn - c = (1-3c)n^2 + (2-3c)n + 1 - c$$

So ??? is now $c(n+1)^3 + \left[(1-3c)n^2 + (2-3c)n + 1 - c\right]$

Can drop [. . .] (through ≤) if it cannot become positive. Happens if $c \ge 1$

b. Show $\Omega(n^3)$

(i) $\sum_{i=1}^{n=1} i^2 = 1 \geq cn^3$ using any constant $0 < c \leq 1$

(ii) Suppose this holds for n:

$$\sum_{i=1}^{n} i^2 \geq cn^3$$

Now go on to $n + 1$ and show that the bound *still holds*

$$\sum_{i=1}^{n+1} i^2 = \sum_{i=1}^{n} i^2 + (n+1)^2$$

$$= \sum_{i=1}^{n} i^2 + n^2 + 2n + 1$$

$$\geq cn^3 + n^2 + 2n + 1$$

$$= ???$$

$$\geq c(n+1)^3$$

The bridging step (???) involves the same algebra as before.

Can drop [. . .] (through ≥) if it cannot become negative. Happens if $0 < c \leq 1/3$

Suppose we attempt to show $\sum_{i=1}^{n} i^2 = \Theta(n^2)$

a. Show $O(n^2)$

(i) $\sum_{i=1}^{n=1} i^2 = 1 \leq cn^2$ using any constant $c \geq 1$

(ii) Suppose this holds for n:

$$\sum_{i=1}^{n} i^2 \leq cn^2$$

Now attempt to go on to $n + 1$.

$$\sum_{i=1}^{n+1} i^2 = \sum_{i=1}^{n} i^2 + (n+1)^2$$

$$= \sum_{i=1}^{n} i^2 + n^2 + 2n + 1$$

$$\leq cn^2 + n^2 + 2n + 1$$

$$= ???$$

$$\leq c(n+1)^2$$

The bridging step (???) separates the bounding term ($c(n+1)^2$) from everything else (x):

$$c(n+1)^2 + x = cn^2 + n^2 + 2n + 1$$

$$x = cn^2 + n^2 + 2n + 1 - cn^2 - 2cn - c = n^2 + (2-2c)n + 1 - c$$

So ??? is now $c(n+1)^2 + \left[n^2 + (2-2c)n + 1 - c\right]$

Can drop [. . .] (through ≤) if it cannot become positive. ***Fails as n grows.***

b. Can still show $\Omega(n^2)$

(i) $\sum_{i=1}^{n=1} i^2 = 1 \geq cn^2$ using any constant $0 < c \leq 1$

(ii) Suppose this holds for n:

$$\sum_{i=1}^{n} i^2 \geq cn^2$$

Now go on to $n + 1$.

$$\sum_{i=1}^{n+1} i^2 = \sum_{i=1}^{n} i^2 + (n+1)^2$$
$$= \sum_{i=1}^{n} i^2 + n^2 + 2n + 1$$
$$\geq cn^2 + n^2 + 2n + 1$$
$$= ???$$
$$\geq c(n+1)^2$$

The bridging step separates the bounding term ($c(n+1)^2$) from everything else (x):

$$c(n+1)^2 + x = cn^2 + n^2 + 2n + 1$$
$$x = cn^2 + n^2 + 2n + 1 - cn^2 - 2cn - c = n^2 + (2-2c)n + 1 - c$$

So ??? is now $c(n+1)^2 + \left[n^2 + (2-2c)n + 1 - c\right]$

Can drop [. . .] (through $\geq$) if it cannot become negative.

Happens if $0 < c \leq 1$ (or for "sufficiently large" n).