CSE 3318 Notes 4: Recurrences

(Last updated 8/1/22 2:59 PM)

CLRS 4.3, 4.4

4.A. Bounding Recurrences Asymptotically

Goal: Take a function $T(n)$ that is defined recursively and find $f(n)$ such that $T(n) \in \Theta(f(n))$.

Need to establish both $T(n) \in O(f(n))$ and $T(n) \in \Omega(f(n))$ (without using the limit theorems).

4.B. Recurrences, Constants, and Evaluating by Brute Expansion

Consider the recurrence:

$$T(n) = T\left(\frac{n}{2}\right) + e = T\left(\frac{n}{2}\right) + \Theta(1)$$

$$T(1) = d = \Theta(1) \quad \text{[or use an arbitrary } n_0\text{, i.e. } T(n_0) = d = \Theta(1)\text{]}$$

Suppose $n = 2^k$:

$$T(1) = d$$
$$T(2) = T(1) + e = d + e$$
$$T(4) = T(2) + e = (d + e) + e = d + 2e$$
$$T(8) = T(4) + e = (d + 2e) + e = d + 3e$$
$$T\left(2^k\right) = T\left(2^{k-1}\right) + e = d + ke = d + e \lg n = \Theta(\log n)$$

Consider the recurrence:

$$T(n) = 2T\left(\frac{n}{2}\right) + en = 2T\left(\frac{n}{2}\right) + \Theta(n)$$

$$T(1) = d = \Theta(1)$$

Suppose $n = 2^k$:

$$T(1) = d$$
$$T(2) = 2T(1) + 2e = 2d + 2e$$
$$T(4) = 2T(2) + 4e = 2(2d + 2e) + 4e = 4d + 8e$$
$$T(8) = 2T(4) + 8e = 2(4d + 8e) + 8e = 8d + 24e$$
$$T(16) = 2T(8) + 16e = 2(8d + 24e) + 16e = 16d + 64e$$
$$T\left(n = 2^k\right) = 2T\left(2^{k-1}\right) + 2^k e = 2\left(2^{k-1}d + 2^{k-1}(k-1)e\right) + 2^k e$$
$$= 2^k d + 2^k ke = nd + en\lg n = \Theta(n \log n)$$

(see `https://ranger.uta.edu/~weems/NOTES3318/notes04.c` for examples of expansion)

Constants *d* and *e* rarely matter . . . certainly not for $T(n) = aT\left(\frac{n}{b}\right) + f(n)$ recurrences

4.C. The Substitution Method for Bounding Recurrences

Method

Guess bound (lower Ω and/or upper O) [On exams the guess will be given to you]

Verify by math induction (solve for constants for some function in asymptotic set)

i) Assume bounding hypothesis works for $k < n$ [Never, ever $k \le n$]

ii) Show bounding hypothesis works for *n* in ***exactly*** the same form as (i).

Refine bound

Example: Binary search recurrence (number of probes)

$$T(n) = T\left(\frac{n}{2}\right) + 1$$

$O(\log n)$

Assume $T(k) \le c \lg k$ for $k < n$. (Note: $\log_a n \in \Theta(\log_b n)$)

$$T\left(\frac{n}{2}\right) \le c\lg\left(\frac{n}{2}\right) = c(\lg n - \lg 2) = c\lg n - c$$
$$T(n) = T\left(\frac{n}{2}\right) + 1$$
$$\le c\lg n - c + 1$$
$$\le c\lg n \text{ if } c \ge 1$$

$\Omega(\log n)$

Assume $T(k) \geq c \lg k$ for $k < n$

$$T\left(\frac{n}{2}\right) \geq c \lg\left(\frac{n}{2}\right) = c(\lg n - \lg 2) = c \lg n - c$$

$$\begin{aligned} T(n) &= T\left(\frac{n}{2}\right) + 1 \\ &\geq c \lg n - c + 1 \\ &\geq c \lg n \text{ if } 0 < c \leq 1 \end{aligned}$$

Example: $T(n) = 2T\left(\frac{n}{2}\right) + dn$

$O(n \log n)$

Assume $T(k) \leq ck \lg k$ for $k < n$

$$T\left(\frac{n}{2}\right) \leq c\frac{n}{2} \lg\left(\frac{n}{2}\right) = c\frac{n}{2}(\lg n - \lg 2) = c\frac{n}{2}\lg n - c\frac{n}{2}$$

$$\begin{aligned} T(n) &= 2T\left(\frac{n}{2}\right) + dn \\ &\leq 2\left(c\frac{n}{2}\lg n - c\frac{n}{2}\right) + dn \\ &= cn \lg n - cn + dn \\ &\leq cn \lg n \text{ if } c \geq d \end{aligned}$$

$\Omega(n \log n)$

Assume $T(k) \geq ck \lg k$ for $k < n$

$$T\left(\frac{n}{2}\right) \geq c\frac{n}{2} \lg\left(\frac{n}{2}\right) = c\frac{n}{2}(\lg n - \lg 2) = c\frac{n}{2}\lg n - c\frac{n}{2}$$

$$\begin{aligned} T(n) &= 2T\left(\frac{n}{2}\right) + dn \\ &\geq 2\left(c\frac{n}{2}\lg n - c\frac{n}{2}\right) + dn \\ &= cn \lg n - cn + dn \\ &\geq cn \lg n \text{ if } 0 < c \leq d \end{aligned}$$

Why is there no basis step for the math induction in these two examples?

Example: $T(n) = 4T\left(\frac{n}{2}\right) + n$

Try $O\left(n^3\right)$ and confirm by math induction:

Assume $T(k) \le ck^3$ for $k < n$

$$T\left(\frac{n}{2}\right) \le c\frac{n^3}{8}$$

$$\begin{aligned}
T(n) &= 4T\left(\frac{n}{2}\right) + n \\
&\le 4c\frac{n^3}{8} + n \\
&= \frac{c}{2}n^3 + n \\
&= cn^3 - \frac{c}{2}n^3 + n \qquad -\frac{c}{2}n^3 + n \le 0 \text{ if } n \text{ is sufficiently large} \\
&\le cn^3
\end{aligned}$$

Improve bound to $O\left(n^2\right)$ and confirm:

Assume $T(k) \le ck^2$ for $k < n$

$$T\left(\frac{n}{2}\right) \le c\frac{n^2}{4}$$

$$T(n) = 4T\left(\frac{n}{2}\right) + n \le 4c\frac{n^2}{4} + n = cn^2 + n \quad \text{STUCK!}$$

$\Omega\left(n^3\right)$ as lower bound:

Assume $T(k) \geq ck^3$ for $k < n$

$$T\left(\frac{n}{2}\right) \geq c\frac{n^3}{8}$$

$$T(n) = 4T\left(\frac{n}{2}\right) + n$$

$$\geq 4c\frac{n^3}{8} + n$$

$$= \frac{c}{2}n^3 + n$$

$$= cn^3 - \frac{c}{2}n^3 + n \qquad -\frac{c}{2}n^3 + n \geq 0\,?$$

$\geq cn^3$ DID NOT PROVE!!!

$\Omega\left(n^2\right)$ as lower bound:

Assume $T(k) \geq ck^2$ for $k < n$

$$T\left(\frac{n}{2}\right) \geq c\frac{n^2}{4}$$

$$T(n) = 4T\left(\frac{n}{2}\right) + n \geq 4c\frac{n^2}{4} + n = cn^2 + n \geq cn^2 \text{ for } 0 < c$$

What's going on? May use either brute expansion (or a recursion tree, see 4.D)

$$T(1) = d$$

$$T(2) = 4T(1) + 2 = 4d + 2$$

$$T(4) = 4T(2) + 4 = 4(4d + 2) + 4 = 16d + 8 + 4 = 16d + 12$$

$$T(8) = 4T(4) + 8 = 4(16d + 12) + 8 = 64d + 48 + 8 = 64d + 56$$

$$T(16) = 4T(8) + 16 = 4(64d + 56) + 16 = 256d + 224 + 16 = 256d + 240$$

Hypothesis: $T(n) = dn^2 + n^2 - n = (d + 1)n^2 - n = cn^2 - n = \Theta\left(n^2\right)$

$O(n^2)$:

Assume $T(k) \le ck^2 - k$ for $k < n$

$$T\left(\frac{n}{2}\right) \le c\frac{n^2}{4} - \frac{n}{2}$$

$$T(n) = 4T\left(\frac{n}{2}\right) + n \le 4\left(c\frac{n^2}{4} - \frac{n}{2}\right) + n = cn^2 - 2n + n = cn^2 - n$$

$\Omega(n^2)$: [This bound was already proven.]

Assume $T(k) \ge ck^2 - k$ for $k < n$

$$T\left(\frac{n}{2}\right) \ge c\frac{n^2}{4} - \frac{n}{2}$$

$$T(n) = 4T\left(\frac{n}{2}\right) + n \ge 4\left(c\frac{n^2}{4} - \frac{n}{2}\right) + n = cn^2 - 2n + n = cn^2 - n$$

Example: $T(n) = T\left(\sqrt{n}\right) + 1$

$O(\log n)$:

Assume $T(k) \le c \lg k$ for $k < n$

$$T\left(\sqrt{n}\right) \le c \lg \sqrt{n} = c\frac{\lg n}{2}$$

$$T(n) = T\left(\sqrt{n}\right) + 1 \le c\frac{\lg n}{2} + 1 = c \lg n - c\frac{\lg n}{2} + 1$$

$$\le c \lg n \text{ for sufficiently large } n$$

$O(\log \log n)$:

Assume $T(k) \le c \lg \lg k$ for $k < n$. (Note: $\log_a \log_a n \in \Theta(\log_b \log_b n)$)

$$T(\sqrt{n}) \le c \lg \lg \sqrt{n} = c \lg \frac{\lg n}{2} = c \lg \lg n - c$$

$$T(n) = T(\sqrt{n}) + 1 \le c \lg \lg n - c + 1$$

$$\le c \lg \lg n \text{ if } c \ge 1$$

Achieving $\Theta(\log \log n)$ seems irrelevant, but . . .

2^n is to n as n is to $\log n$ . . . and as $\log n$ is to $\log \log n$.

$\Omega(\log \log n)$:

Assume $T(k) \ge c \lg \lg k$ for $k < n$

$$T(\sqrt{n}) \ge c \lg \lg \sqrt{n} = c \lg \frac{\lg n}{2} = c \lg \lg n - c$$

$$T(n) = T(\sqrt{n}) + 1 \ge c \lg \lg n - c + 1$$

$$\ge c \lg \lg n \text{ if } 0 < c \le 1$$

Example: $T(n) = 2T\left(\frac{n}{2}\right) + n^3$

$O(n^3)$:

Assume $T(k) \le ck^3$ for $k < n$

$$T\left(\frac{n}{2}\right) \le c \frac{n^3}{8}$$

$$T(n) = 2T\left(\frac{n}{2}\right) + n^3 \le 2c \frac{n^3}{8} + n^3 = c \frac{n^3}{4} + n^3$$

$$= cn^3 - \frac{3}{4} cn^3 + n^3$$

$$\le cn^3 \text{ if } c \ge \frac{4}{3}$$

$\Omega\left(n^3\right)$:

Assume $T(k) \geq ck^3$ for $k < n$

$$T\left(\frac{n}{2}\right) \geq c\frac{n^3}{8}$$

$$T(n) = 2T\left(\frac{n}{2}\right) + n^3 \geq 2c\frac{n^3}{8} + n^3 = c\frac{n^3}{4} + n^3$$

$$= cn^3 - \frac{3}{4}cn^3 + n^3$$

$$\geq cn^3 \text{ if } 0 < c \leq \frac{4}{3}$$

Alternative proof of $O\left(n^3\right)$:

Assume $T(k) \leq ck^3$ for $k < n$

$$T\left(\frac{n}{2}\right) \leq c\frac{n^3}{8}$$

$$T(n) = 2T\left(\frac{n}{2}\right) + n^3 \leq 2c\frac{n^3}{8} + n^3 = c\frac{n^3}{4} + n^3$$

$$= n^3\left(\frac{c}{4} + 1\right)$$

$$\leq cn^3 \text{ if } c \geq \frac{4}{3} \text{ which is derived by: } \quad \frac{c}{4} + 1 \leq c$$

$$1 \leq \frac{3}{4}c$$

$$\frac{4}{3} \leq c$$

Example: $T(n) = 3T\left(\frac{n}{3}\right) + 2$

$O(n)$:

Assume $T(k) \leq ck$ for $k < n$

$$T\left(\frac{n}{3}\right) \leq \frac{cn}{3}$$

$$T(n) = 3T\left(\frac{n}{3}\right) + 2 \leq 3\frac{cn}{3} + 2 = cn + 2 \text{ stuck!}$$

$\Omega(n)$:

Assume $T(k) \geq ck$ for $k < n$

$$T\left(\frac{n}{3}\right) \geq \frac{cn}{3}$$

$$T(n) = 3T\left(\frac{n}{3}\right) + 2 \geq 3\frac{cn}{3} + 2 = cn + 2 \geq cn$$

Examine a few cases:

$$T(1) = d$$
$$T(3) = 3T(1) + 2 = 3d + 2$$
$$T(9) = 3T(3) + 2 = 3(3d + 2) + 2 = 9d + 6 + 2 = 9d + 8$$
$$T(27) = 3T(9) + 2 = 3(9d + 8) + 2 = 27d + 24 + 2 = 27d + 26$$

Hypothesis: $T(n) = nd + n - 1 = (d + 1)n - 1 = cn - 1 = \Theta(n)$

New try for $O(n)$:

Assume $T(k) \leq ck - 1$ for $k < n$

$$T\left(\frac{n}{3}\right) \leq \frac{cn}{3} - 1$$

$$T(n) = 3T\left(\frac{n}{3}\right) + 2 \leq 3\left(\frac{cn}{3} - 1\right) + 2 = cn - 3 + 2 = cn - 1$$

4.D. Recursion Tree Method

Concepts

Draw tree – either for a particular n or for general case

Fan-out

Sub-problem sizes

Number of levels in tree

Number of leaves

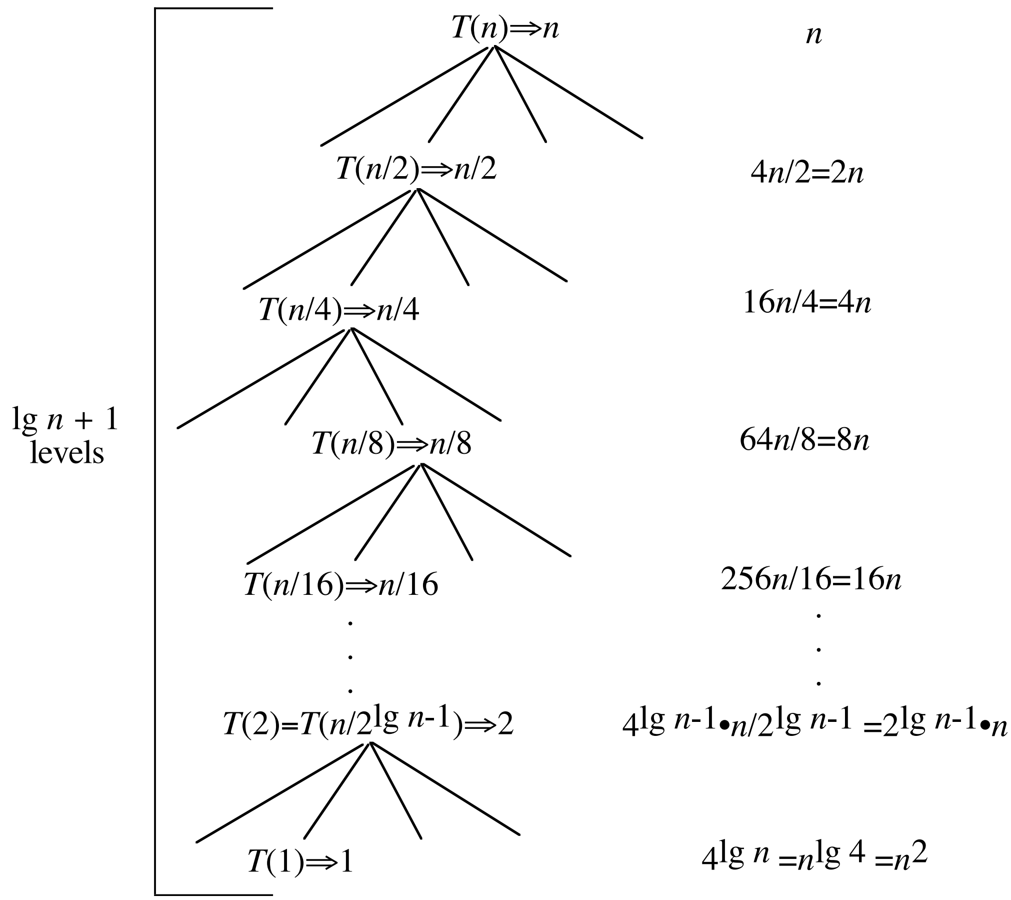
(Analysis at parents-of-leaves level is OPTIONAL!)

Assign (non-recursive) contribution of a node (sub-problem) in each level

Compute total (non-recursive) contribution across each level [different from brute expansion]

Usually complete by evaluating a summation – Go directly for Θ bound, not separate O and Ω.

Example: $T(n) = 4T\left(\frac{n}{2}\right) + n$



Note use of the identity $b^{\log_a d} = d^{\log_a b}$. (CLRS, p. 66)

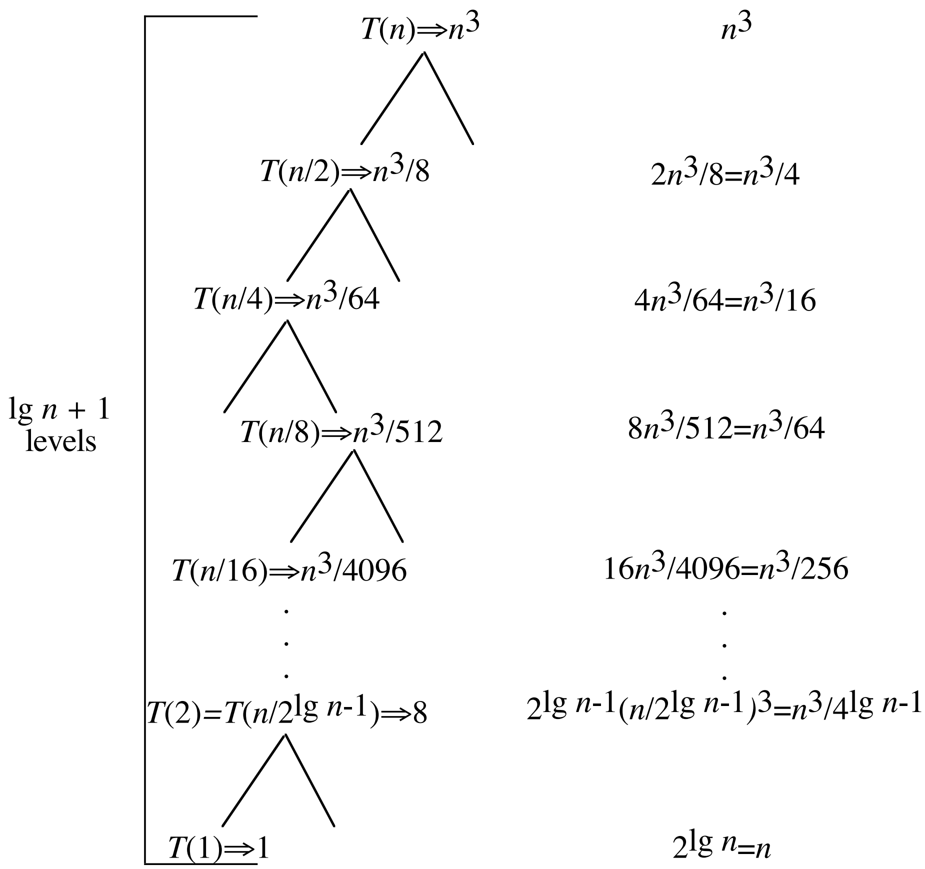
Using definite geometric sum formula:

$$n\sum_{k=0}^{\lg n-1} 2^k + n^2 = n\frac{2^{\lg n} - 1}{2-1} + n^2 \qquad \text{Using } \sum_{k=0}^{t} x^k = \frac{x^{t+1}-1}{x-1} \qquad x \neq 1$$

$$= n(n-1) + n^2$$

$$= n^2 - n + n^2 = \Theta\left(n^2\right)$$

Example: $T(n) = 2T\left(\frac{n}{2}\right) + n^3$



Using indefinite geometric sum formula:

$$n^3 \sum_{k=0}^{\lg n-1} \frac{1}{4^k} + n \le n^3 \sum_{k=0}^{\infty} \frac{1}{4^k} + n$$

$$= n^3 \frac{1}{1-\frac{1}{4}} + n \qquad \text{From } \sum_{k=0}^{\infty} x^k = \frac{1}{1-x} \qquad 0 < x < 1$$

$$= \frac{4}{3} n^3 + n = \mathrm{O}(n^3)$$

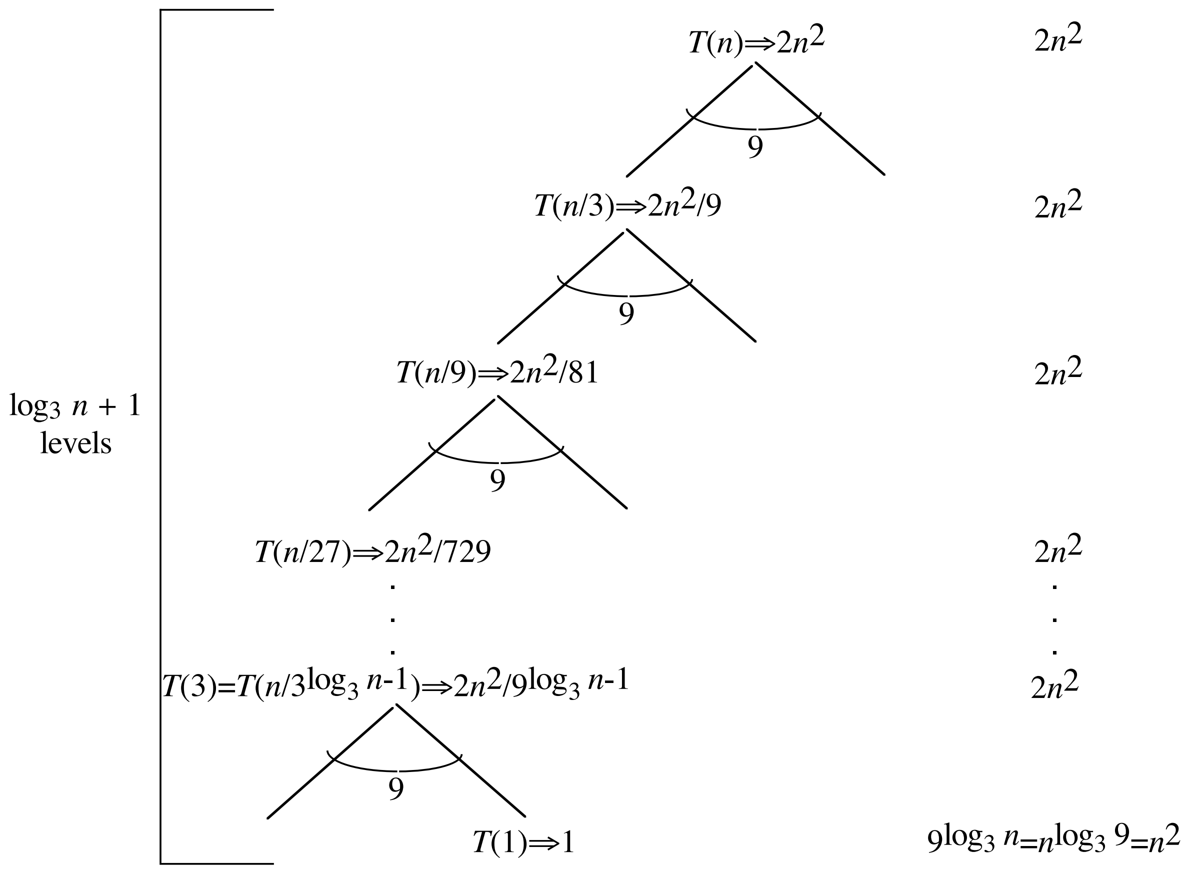
Using definite geometric sum formula:

$$n^3 \sum_{k=0}^{\lg n - 1} \frac{1}{4^k} + n = n^3 \frac{\left(\frac{1}{4}\right)^{\lg n} - 1}{\frac{1}{4} - 1} + n \qquad \text{Using } \sum_{k=0}^{t} x^k = \frac{x^{t+1} - 1}{x - 1} \qquad x \neq 1$$

$$= n^3 \frac{n^{-2} - 1}{-\frac{3}{4}} + n = n^3 \frac{1 - n^{-2}}{\frac{3}{4}} + n = \frac{4}{3} n^3 \left(1 - \frac{1}{n^2}\right) + n$$

$$= \Theta\left(n^3\right)$$

Example: $T(n) = 9T\left(\frac{n}{3}\right) + 2n^2$



$$T(n) = 2n^2 \log_3 n + n^2 = \Theta\left(n^2 \log n\right)$$