CSE 3318 Notes 15: Shortest Paths

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CLRS 22.3, 23.2

15.A. CONCEPTS

(Aside: `https://dl-acm-org.ezproxy.uta.edu/doi/10.1145/2530531`)

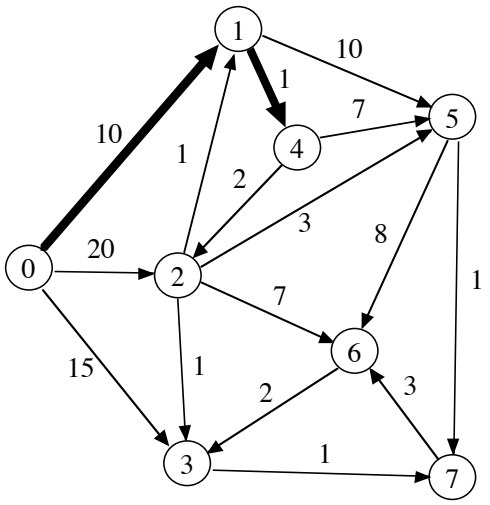
Input:

> Directed graph with *non-negative* edge weights (stored as adj. matrix for Floyd-Warshall)
> Dijkstra – source vertex

Output:

> Dijkstra – tree that gives a shortest path from source to each vertex
> Floyd-Warshall – shortest path between each pair of vertices ("all-pairs") as matrix

15.B. DIJKSTRA'S ALGORITHM – three versions



0	1	2	3	4	5	6	7
0(-)	∞	∞	∞	∞	∞	∞	∞
*	10(0)	20(0)	15(0)				
	*			11(1)	20(1)		
		13(4)		*	18(4)		
		*	14(2)		16(2)	20(2)	
			*				15(3)
						18(7)	*
					*		
						*	

Similar to Prim's MST:

S = vertices whose shortest path is known (initially just the source)

Length of path
Predecessor (vertex) on path (AKA shortest path tree)

T = vertices whose shortest path is not known

Each phase moves a T vertex to S by virtue of that vertex having the shortest path among all T vertices.

Third version may be viewed as being BFS with the FIFO queue replaced by a priority queue.

1. "Memoryless" – Only saves shortest path tree and current partition.
   (`https://ranger.uta.edu/~weems/NOTES3318/dijkstraMemoryless.c`)

Place desired source vertex $x \in V$ in S
$T = V - \{x\}$
x.distance = 0
x.pred = (-1)
while $T \neq \varnothing$
    Find the edge (s, t) over all $t \in T$ and all $s \in S$ with minimum value for s.distance + weight(s, t)
        (i.e. scan adj. list for each s)
    t.distance = s.distance + weight(s, t)
    t.pred = s
    $T = T - \{t\}$
    $S = S \cup \{t\}$

Since no substantial data structures are used, this takes $\Theta(EV)$ time.

2. Maintains T-table that provides the predecessor vertex in S for each vertex $t \in T$ to give the shortest possible path through S to t. (`https://ranger.uta.edu/~weems/NOTES3318/dijkstraTable.c`)

Eliminates scanning all S adjacency lists in every phase, but still scans the list of the last vertex moved from T to S.

Place desired source vertex $x \in V$ in S
$T = V - \{x\}$
x.distance = 0
x.pred = (-1)
for each $t \in T$
    Initialize t.distance with weight of (x, t) (or ∞ if non-existent) and t.pred = x

```
while T ≠ ∅
        Scan T entries to find vertex t with minimum value for t.distance
        T = T – {t}
        S = S ∪ {t}
        for each vertex x in adjacency list of t (i.e. (t, x))
                if x ∈ T and t.distance + weight(t, x) < x.distance
                        x.distance = t.distance + weight(t, x)
                        x.pred = t
```

Analysis:

Initializing the T-table takes $\Theta(V)$.
Scans of T-table entries contribute $\Theta(V^2)$.
Traversals of adjacency lists contribute $\Theta(E)$.
$\Theta\left(V^2 + E\right) = \Theta(V^2)$ overall worst-case.

3. Replace T-table by a min-heap.
(`https://ranger.uta.edu/~weems/NOTES3318/dijkstraHeap.cpp`)

The time for updating distances and predecessors increases, but the time for selection of the next vertex to move from T to S improves.

```
Place desired source vertex x ∈ V in S
T = V – {x}
x.distance = 0
x.pred = (-1)
for each t ∈ T
        Initialize T-heap with weight (as the priority) of (x, t) (or ∞ if non-existent) and t.pred = x
minHeapInit(T-heap)  // a fixDown at each parent node in heap
while T ≠ ∅
        Use heapExtractMin /* fixDown */ to obtain T-heap entry with minimum t.distance
        T = T – {t}
        S = S ∪ {t}
        for each vertex x in adjacency list of t (i.e. (t, x))
                if x ∈ T and t.distance + weight(t, x) < x.distance
                        x.distance = t.distance + weight(t, x)
                        x.pred = t
                        minHeapChange(T-heap)  // fixUp
```

Analysis:

Initializing the T-heap takes $\Theta(V)$.
Total cost for `heapExtractMin`s is $\Theta(V \log V)$.
Traversals of adjacency lists and `minHeapChange`s contribute $\Theta(E \log V)$.
$\Theta\left(E \log V\right)$ overall worst-case, since $E > V$.

Which version is the fastest?

	Theory	Sparse $\left(E = \mathrm{O}(V)\right)$	Dense $\left(E = \Omega\left(V^2\right)\right)$
1.	$\Theta(EV)$	$\Theta\left(V^2\right)$	$\Theta\left(V^3\right)$
2.	$\Theta(\mathrm{V}^2)$	$\Theta\left(V^2\right)$	$\Theta\left(V^2\right)$
3.	$\Theta(E \log V)$	$\Theta(V \log V)$	$\Theta\left(V^2 \log V\right)$

15.C. FLOYD-WARSHALL ALGORITHM

Based on adjacency matrices. Will examine three versions:

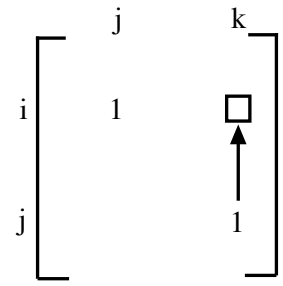
Warshall's Algorithm – After $\Theta(\mathrm{V}^3)$ preprocessing, processes each path *existence* query in $\Theta(1)$ time.

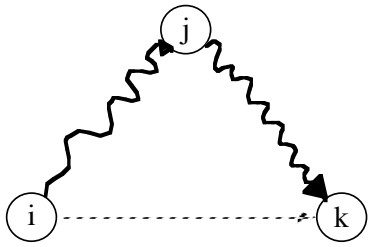
Warshall's Algorithm with Successors (or predecessors or transitive vertices) - After $\Theta(\mathrm{V}^3)$ preprocessing, provides a path in response to a path existence query in O(V) time (similar to dynamic programming backtrace).

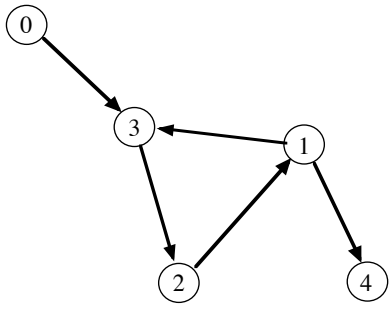
Floyd-Warshall Algorithm (with Successors) - After $\Theta(\mathrm{V}^3)$ preprocessing, provides each *shortest* path in O(V) time.

Warshall's Algorithm:

```
for (j=0; j<V; j++)
  for (i=0; i<V; i++)
    if (A[i][j])
      for (k=0; k<V; k++)
        if (A[j][k])
          A[i][k]=1;
```

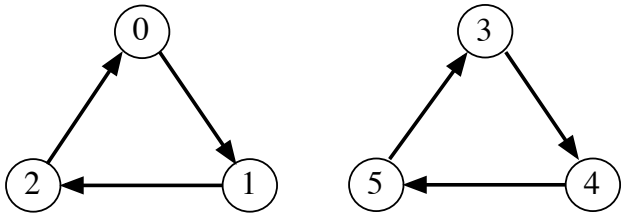






	0	1	2	3	4
0				1	
1				1	1
2		1			
3			1		
4					

If zero-edge paths are useful for an application (i.e. reflexive, self-loops), the diagonal may be all ones.



	0	1	2	3	4	5
0		1				
1			1			
2	1					
3					1	
4						1
5				1		

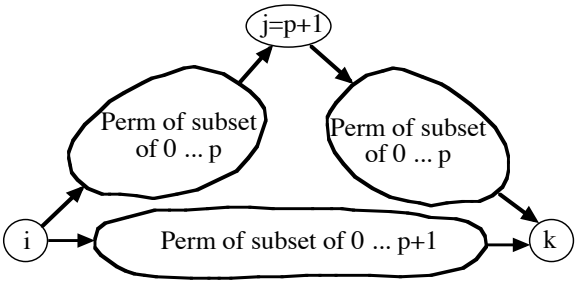
Why does it work?

a. *Correct* in use of transitivity.

b. Is it *complete*?

When	Paths That Can Be Detected
Before j=0	$x \to y$
After j=0	$x \to 0 \to y$
After j=1	$x \to 1 \to y$ $x \to 0 \to 1 \to y$ $x \to 1 \to 0 \to y$
After j=2	$x \to 2 \to y$ $x \to 0 \to 2 \to y$ $x \to 1 \to 2 \to y$ $x \to 2 \to 0 \to y$ $x \to 2 \to 1 \to y$ $x \to 0 \to 1 \to 2 \to y$ $x \to 0 \to 2 \to 1 \to y$ $x \to 1 \to 0 \to 2 \to y$ $x \to 1 \to 2 \to 0 \to y$ $x \to 2 \to 0 \to 1 \to y$ $x \to 2 \to 1 \to 0 \to y$
. . .	. . .
After j=p	$x \to$ Permutation of *subset* of $0 \ldots p \to y$
After j=V-1	ALL PATHS

Math. Induction:

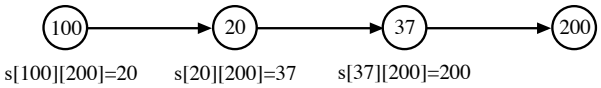


Suppose that there is only one path from vertex 5 to vertex 10 in a directed graph:
$5 \rightarrow 7 \rightarrow 4 \rightarrow 3 \rightarrow 2 \rightarrow 10$. During the scan of which column will Warshall's algorithm record the presence of this path?

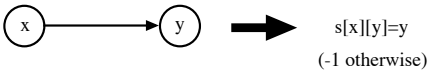
Warshall's Algorithm with Successors

Successor Matrix

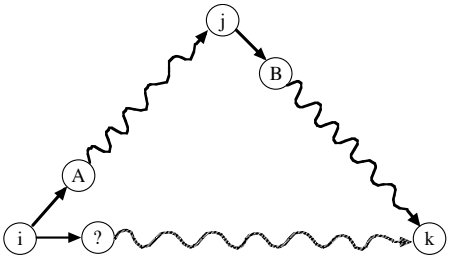
Buc-ee's directions:



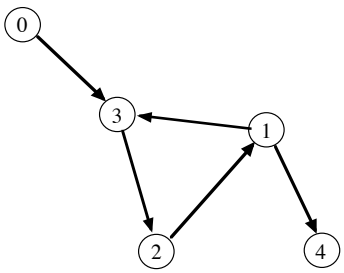
Initialize:



Warshall Matrix Update:



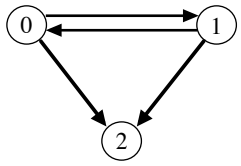
succ[i][j] = A succ[j][k] = B succ[i][k] = ?



	0	1	2	3	4
0				3	
1				3	4
2		1			
3			2		
4					

```
for (j=0; j<V; j++)
  for (i=0; i<V; i++)
    if (s[i][j] != (-1))
      for (k=0; k<V; k++)
        if (succ[i][k]==(-1) && succ[j][k]!=(-1))
          succ[i][k] = succ[i][j];
```

Suppose code in box is removed for this graph:



Complete Example (https://ranger.uta.edu/~weems/NOTES3318/warshall.c) saving paths using successors:

	0	1	2	3	4
0	-1	-1	-1	3	-1
1	-1	-1	-1	3	4
2	-1	1	-1	-1	-1
3	-1	-1	2	-1	-1
4	-1	-1	-1	-1	-1

	0	1	2	3	4
0	-1	-1	-1	3	-1
1	-1	-1	-1	3	4
2	-1	1	-1	1	1
3	-1	2	2	2	2
4	-1	-1	-1	-1	-1

	0	1	2	3	4
0	-1	-1	-1	3	-1
1	-1	-1	-1	3	4
2	-1	1	-1	-1	-1
3	-1	-1	2	-1	-1
4	-1	-1	-1	-1	-1

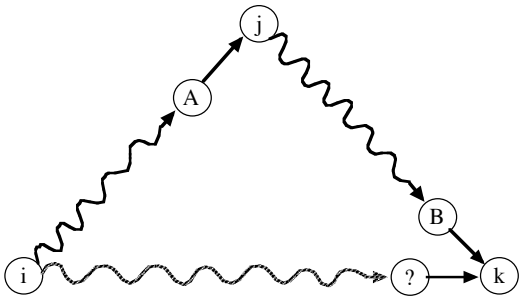
	0	1	2	3	4
0	-1	3	3	3	3
1	-1	3	3	3	4
2	-1	1	1	1	1
3	-1	2	2	2	2
4	-1	-1	-1	-1	-1

	0	1	2	3	4
0	-1	-1	-1	3	-1
1	-1	-1	-1	3	4
2	-1	1	-1	1	1
3	-1	-1	2	-1	-1
4	-1	-1	-1	-1	-1

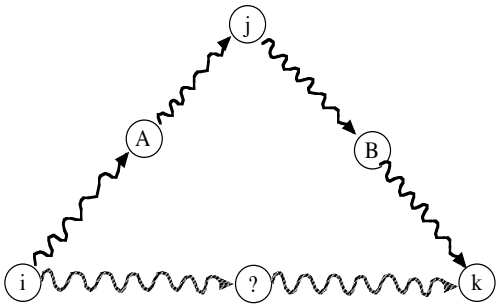
	0	1	2	3	4
0	-1	3	3	3	3
1	-1	3	3	3	4
2	-1	1	1	1	1
3	-1	2	2	2	2
4	-1	-1	-1	-1	-1

Other ways to save path information:

Predecessors (`warshallPred.c`)



Transitive/Intermediate/Column (`warshallCol.c`)

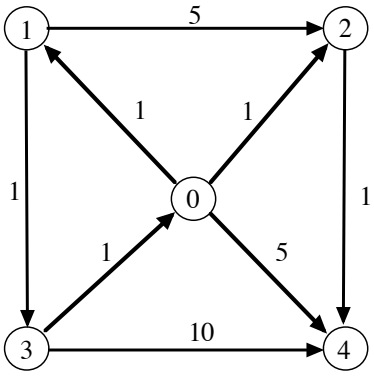



```
for (j=0;j<n;j++)
  for (i=0;i<n;i++)
    if (pred[i][j]!=(-1))
      for (k=0;k<n;k++)
        if (pred[i][k]==(-1) && pred[j][k]!=(-1))
          pred[i][k]=pred[j][k];
```

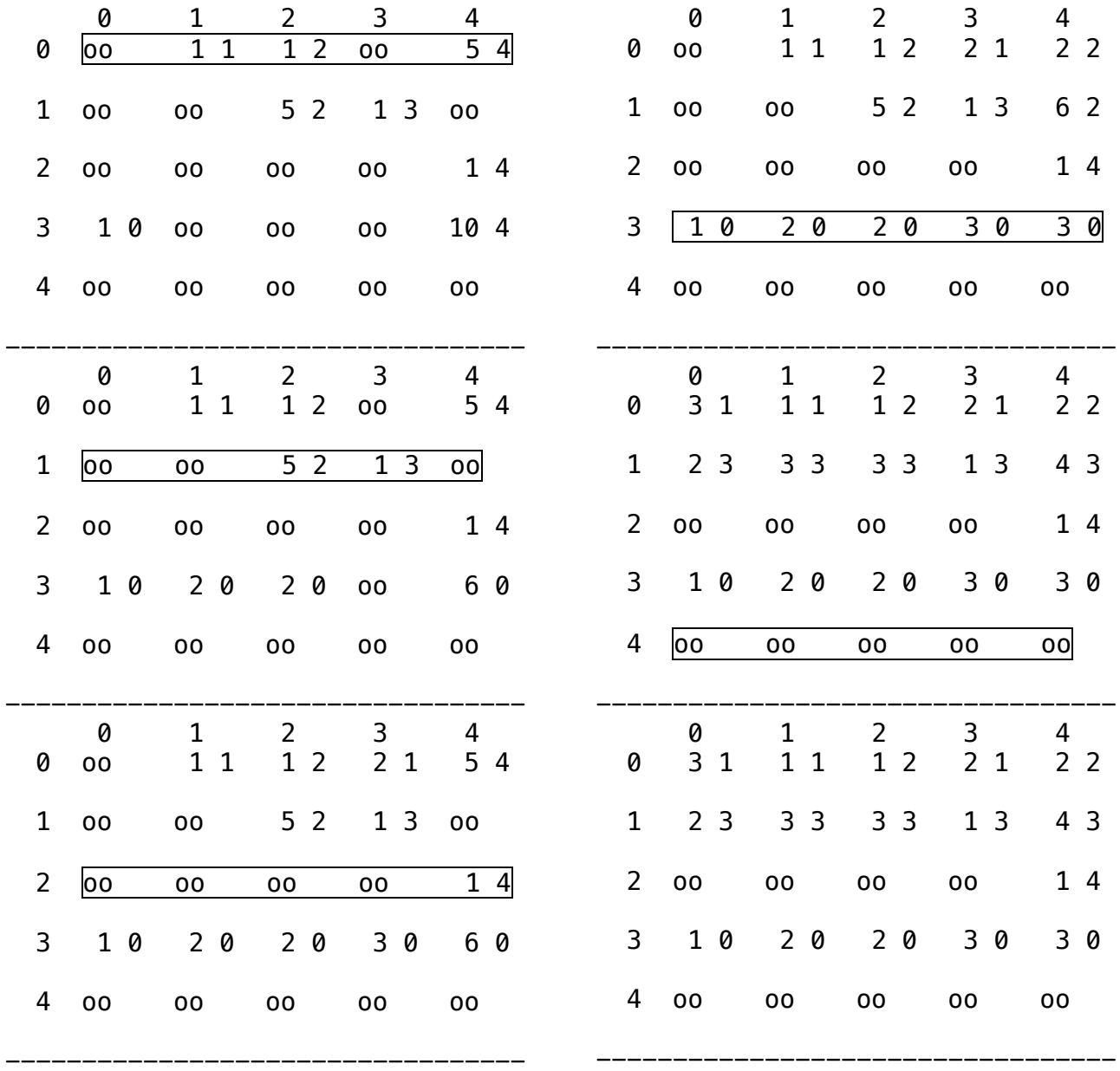
Floyd-Warshall Algorithm with Successors (`https://ranger.uta.edu/~weems/NOTES3318/floydWarshall.c`)

After j = p has been processed, the *shortest path* from each x to each y that uses *only* vertices in 0 . . . p as intermediate vertices is recorded in matrix.

```
for (j=0;j<n;j++)
{
  for (i=0;i<n;i++)
    if (dist[i][j]<oo)
      for (k=0;k<n;k++)
        if (dist[j][k]<oo)
        {
          newDist=dist[i][j]+dist[j][k];
          if (newDist<dist[i][k])
          {
            dist[i][k]=newDist;
            succ[i][k]=succ[i][j];
          }
        }
}
```



	0	1	2	3	4
0		1	1		5
1			5	1	
2					1
3	1				10
4					



Note: In this example, zero-edge paths are not considered.