On-line Algorithm – Homework2

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Problem: Exercise 10.4

- Problem Description:

  Prove the following 2-server algorithm **ALG** is O(1)-competitive in any Euclidean Space.

  ALG: After serving each request, label the server at the request s_1 and the other server s_2 (if both servers are at the request, break tie arbitrarily). Consider the next request r and set $b = d(s_1,r)$. If $d(s_2,r) < 3b$, serve r with s_2. Otherwise, serve r with s_1 and also move s_2 a distance $3b$ toward r.

- We'll consider the following three cases using potential function method:

  Case 1 : only one server is moving ($d(s_2, r) < 3b$)
  Case 2 : two servers are moving ($d(s_2, r) \geq 3b$)
  Case 3 : from two server movement to one server movement ($d(s_2, r) \geq 3b \rightarrow d(s_2, r) < 3b$)

Potential Function Method

- Important tool for analyzing the competitiveness of an on-line algorithm (ALG) in terms of an optimal algorithm (OPT)

- Maps the current configuration of ALG and OPT to a nonnegative value $\Phi \geq 0$

- To prove ALG is c-competitive, find a potential function satisfying the following condition (Interleaving Moves)

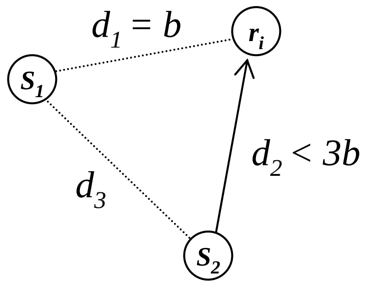
if only ALG moves during event r_i and pays the actual cost x for this move, then $\Delta\Phi = \Phi_i - \Phi_{i-1} \leq -x$

- Defining Potential Function :

$$\Phi = \alpha \cdot M_{\min} + \beta \cdot d(s_1, s_2),\ \alpha=1 \text{ and } \beta=3$$

where $M_{\min}$ is a minimum weight matching between ALG and OPT and $d(s_1, s_2)$ is a distance between two servers after processing i request.

Case 1: one server is moving ($d(s_2, r) < 3b$)



$ALG(r_i) = d_2$ **: actual cost**

$\Delta M_{\min} = -d_2 \rightarrow s_2$ **: change in minimum weight matching**

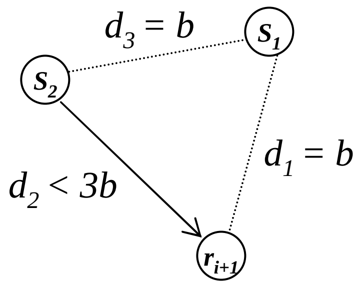
$= 0 \rightarrow s_1$

$\Delta d(s_1, s_2) = d_1 - d_3 = 0$ **: change in $d(S_1, S_2)$ (refer to the bottom figure on the left)**

Thus, $\Delta\Phi \le \alpha \cdot (0 - d_2) + \beta \cdot (d_1 - d_3)$

Since $\Delta\Phi \le -ALG(r_i)$,

$$\alpha \cdot (0 - d_2) + \beta \cdot (d_1 - d_3) \le -ALG(r_i)$$



- For case 3 ($d(s2, r) \ge 3b \rightarrow d(s2, r) < 3b$) : when S_2 processes the request r_i, the change in the $d(S_1, S_2)$-component of Φ is affected by only the distance between S_1 and S_2 due to the fixed distance (=b) between S_1 and r_i (refer to the top figure on the left). Using the triangle inequality: $d_3 \le b + d_2$ and the condition: $d_2 < 3b$, we can conclude that:

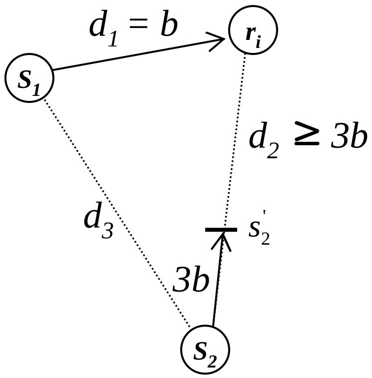
$$0 \le d_3 < 4b$$

Based on this (i.e., if we take the maximum of d_3), the maximum change in the $d(S_1, S_2)$-component is - d_2, That is:

$$\Delta d(s_1, s_2) = b - d_3 \le -d_2$$

Thus, $\alpha \cdot (0 - d_2) + \beta \cdot -d_2 \le -ALG(r_i)$

Case 2:– two servers are moving ($d(s_2, r) \geq 3b$)



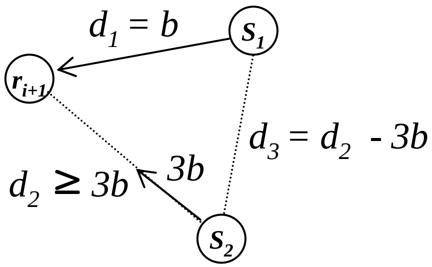
$$ALG(r_i) = d_1 + 3b$$: actual cost

$$\Delta M_{\min} = -d_1 \rightarrow s_1$$
$$= 3b \rightarrow s_2$$: change in minimum weight matching

$$d(s_2', r_i) = d_2 - 3b \leq d_1 + d_3 - 3b$$
$$\leq d_3 - 2b$$

based on *Triangle inequality*
$d_2 \leq d_1 + d_3$

$$\Delta d(s_1, s_2) = d(s_2', r_i) - d_3 \leq d_3 - 2b - d_3$$
$$\leq -2b$$: change in $d(s_1, s_2)$



Thus, $\Delta\Phi \leq \alpha \cdot (3b - b) + \beta \cdot -2b$

Since $\Delta\Phi \leq -ALG(r_i)$,

$$\alpha \cdot (3b - b) + \beta \cdot -2b \leq -ALG(r_i)$$

ALG is O(1)-competitive in any Euclidean Space

- All of the three case are satisfied with $\alpha=1$ and $\beta=3$.

For case 1	$\alpha\cdot(-d_2+0)+\beta\cdot(0)\le -ALG(r_i)$ $-d_2+d_2+3\cdot(0)\le 0$	$\Delta d(s_1,s_2)=0$ $ALG(r_i)=d_2$
For case 2	$\alpha\cdot(3b-b)+\beta\cdot -2b\le -ALG(r_i)$ $\alpha\cdot 2b-\beta\cdot 2b\le -4b$ $2b(\alpha-\beta)+4b\le 0$	$ALG(r_i)=d_1+3b$
For case 3	$\alpha\cdot(-d_2+0)+\beta\cdot -d_2\le -ALG(r_i)$ $-d_2+d_2-d_2\le 0$	$\Delta d(s_1,s_2)\le -d_2$ $ALG(r_i)=d_2$

- Therefore,

  “ALG is O(1)-competitive in any Euclidean Space”.