Exercise 2.4 Consider the following generalization of RMTF. For any real $p \in [0,1]$, let RMTFp be the algorithm that, upon a request for an item x, moves x to the front with probability p. Generalize the lower bound to RMTFp for each $p \in (0,)]$.

**Algorithm RMTF:** Upon a request for an item x, move x to the front with probability 1/2.

RMTF has a lower bound $2-\varepsilon$.

**Algorithm RMTFp:** Upon a request for an item x, move x to the front with probability p.

**We claim that RMTFp has a lower bound $\frac{1}{p}-\varepsilon$ for $p \in (0,1)$.**

**Proof:**

We describe a nemesis request sequence showing that for any given $\varepsilon$, there exists a sufficiently large list length l such that $\tilde{R}(RMTF_p) > \frac{1}{p}-\varepsilon$. Let $\alpha$ and $\varepsilon$ be given, Assume a list of $l = l(\varepsilon)$ elements initially organized as $< x_1, x_2, \cdots, x_l >$ with $x_1$ at the front. Let k be some integer whose value will be determined, and consider the following request sequence $\sigma$:

$$\sigma = (x_l)^k, (x_{l-1})^k, \cdots, (x_1)^k.$$

For large k, with high probability, algorithm RMTFp will move $x_i$ to the front while RMTF services the segment $(x_i)^k$. On average, $x_i$ is moved to the front at the $(\frac{1}{p})^{th}$ request. This is proved as following:

If one element can be moved to the front with probability p, then

| Number of times to move the element to the front, | probability |
|---|---|
| 1 | p |
| 2 | (1-p)p |
| 3 | (1-p)^2p |
| … | … |

expectation of number of accesses to move the element to the front is

$$1.p + 2.(1-p)p + 3(1-p)^2 p + \cdots k.(1-p)^{k-1} p + \cdots = \frac{1}{p}$$

The expected cost for MRTF p to server the segment $(x_l)^k$ is at least $\frac{1}{p}l+(k-\frac{1}{p})$ because the cost to server each of the first $\frac{1}{p}$ elements is l, after that $x_l$ is moved to the front. The cost to serve the rest of $(k-\frac{1}{p})$.

Therefore, the total expected cost for MRTF to serve request sequence $\sigma$ is at least

$$RMTF_p(\sigma) > l(\frac{1}{p}l+k-\frac{1}{p}) = l(\frac{1}{p}l+\frac{1}{p}k+\frac{1}{p}-(1-\frac{1}{p})k)$$
$$= l\frac{1}{p}(l+k-1)-(1-\frac{1}{p})kl$$

On the other hand,

$$MTF(\sigma) = l(l+k-1)$$

Therefore,

$$\tilde{R}(RMTF_p) = \frac{RMTF_p(\sigma)}{OPT(\sigma)} \geq \frac{RMTF_p(\sigma)}{MTF(\sigma)} > \frac{l\frac{1}{p}(l+k-1)-(1-\frac{1}{p})kl}{l(l+k-1)} = \frac{1}{p} - \frac{(1-\frac{1}{p})k}{l+k-1}$$

As long as we choose k such that

$\frac{(1-\frac{1}{p})k}{l+k-1} \geq \varepsilon$ , which implies $k \geq \frac{\varepsilon p(l-1)}{(1-\varepsilon)p-1}$.

We have

$$\tilde{R}(RMTF_p) > \frac{1}{p} - \varepsilon .$$ □